EPFL

Disal

# Signals, Instruments, and Systems – W5

# Introduction to Signal Processing – Bode Plots, Z-Transform, Digital Filters, Order and Type of Filters

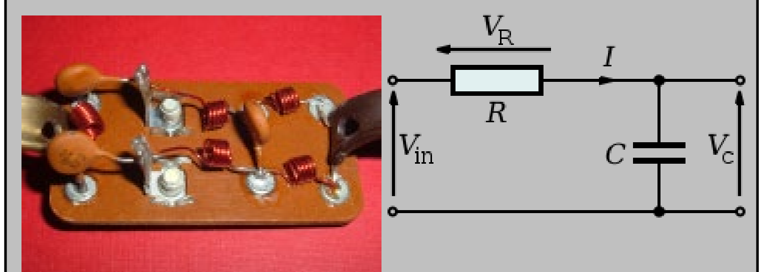
# Outline

- Bode plots
- Z-transform and frequency responses/transfer functions of time-discrete systems
- Digital filters in time and frequency domains
- FIR and IIR filters
- Filter order and type

# Bode Plots

# Transfer Functions of Analog Filters

Circuit




**Laplace Transf.**

$$H(s) = \frac{v_c}{v_{in}} = \frac{\overbrace{1}^{\text{Numerator}}}{\underbrace{1 + RCs}_{\text{Denominator}}}$$

Numerator

Denominator

See also W4, s. 40

# From Transfer Functions to Bode Plots

Assume transfer function: $H(s) = \frac{Num(s)}{Den(s)} = \text{A} \prod \frac{(s - x_n)^{a_n}}{(s - y_n)^{b_n}}$

where $x_n$, $y_n$ constants, $a_n$, $b_n > 0$

Frequency response: $\text{s} = i\omega$

See also W4, s. 20 and 41

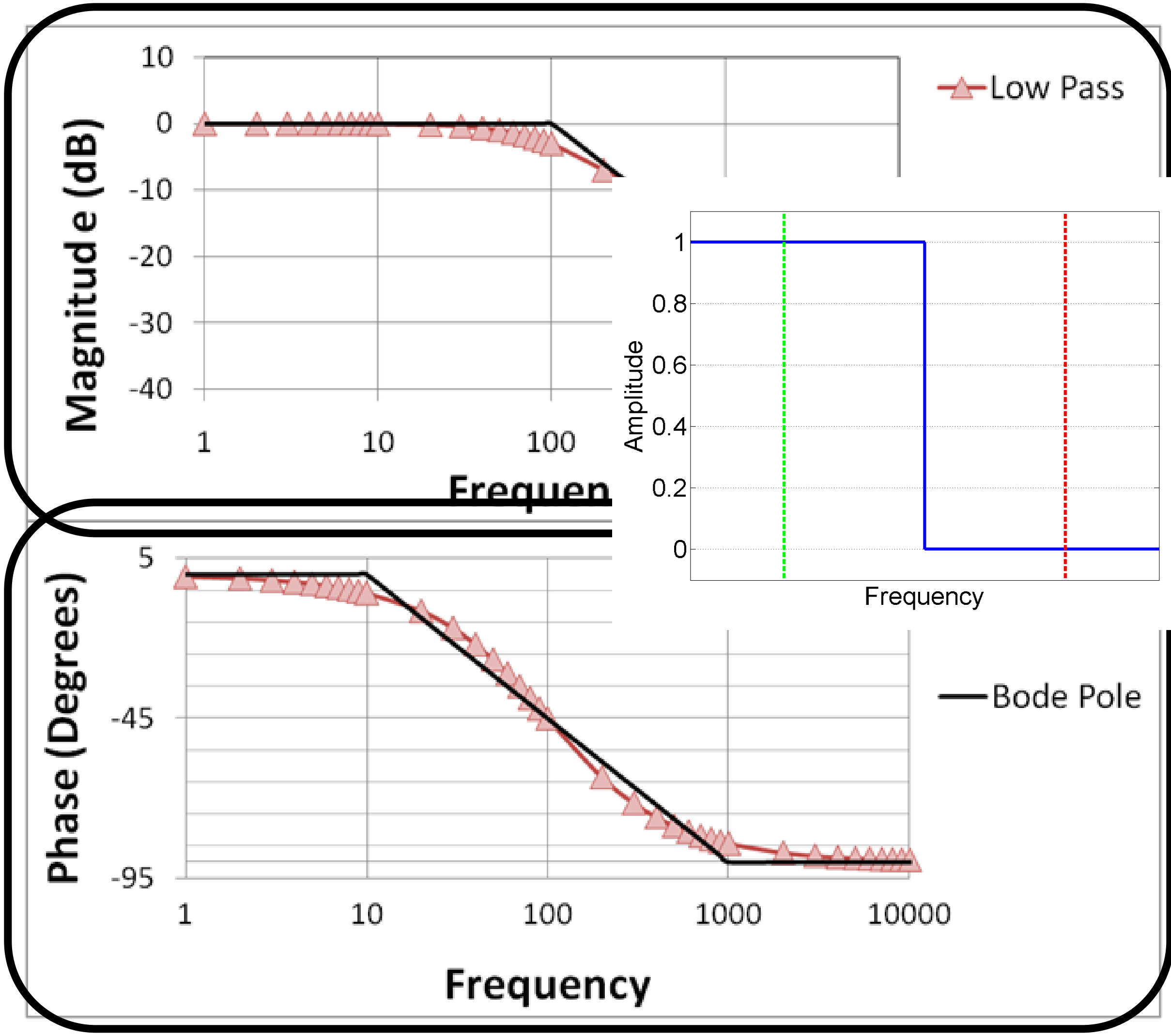
Bode **magnitude**: $|H(s = i\omega)| = |H(i\omega)| = |H(\omega)|$

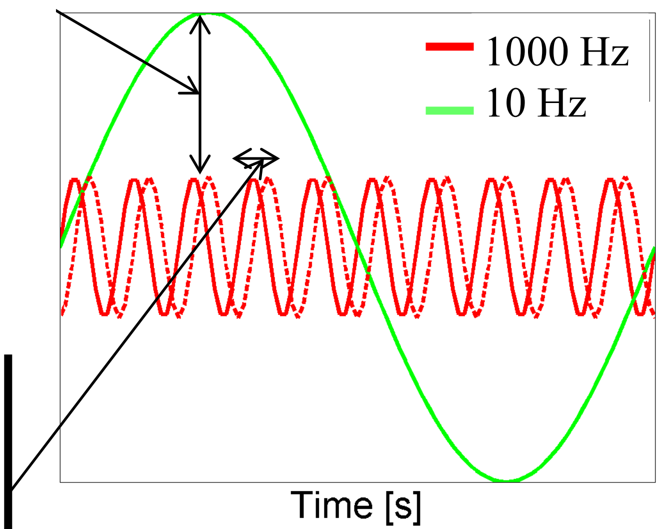
Bode **phase**: $\measuredangle H(s = i\omega) = \measuredangle H(i\omega) = \measuredangle H(\omega)$

**Zero** (numerator = 0): every value of s where $\omega = |x_n|$

**Pole** (denominator = 0): every value of s where $\omega = |y_n|$

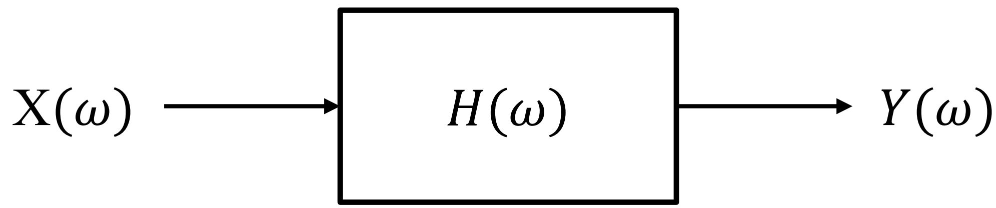
# Bode Plot: The Example of a Low-Pass Filter




**not to scale !**

# Bode Plots – Why handy?




**Frequency domain**

$$Y(\omega) = H(\omega)X(\omega)$$

$$|Y(\omega)| = |H(\omega)||X(\omega)|$$ Amplitude

$$\measuredangle Y(\omega) = \measuredangle H(\omega) + \measuredangle X(\omega)$$ Phase

$$\log|Y(\omega)| = \log|H(\omega)| + \log|X(\omega)|$$

Note: this is valid also for cascaded blocks of LTI systems (e.g., cascaded filters)

# Bode Plot – Why only positive and log scale for ω as well?

– If impulse response h(t) real, then $|H(\omega)|$ **even** function of ω and $\measuredangle H(\omega)$ **odd** function of ω

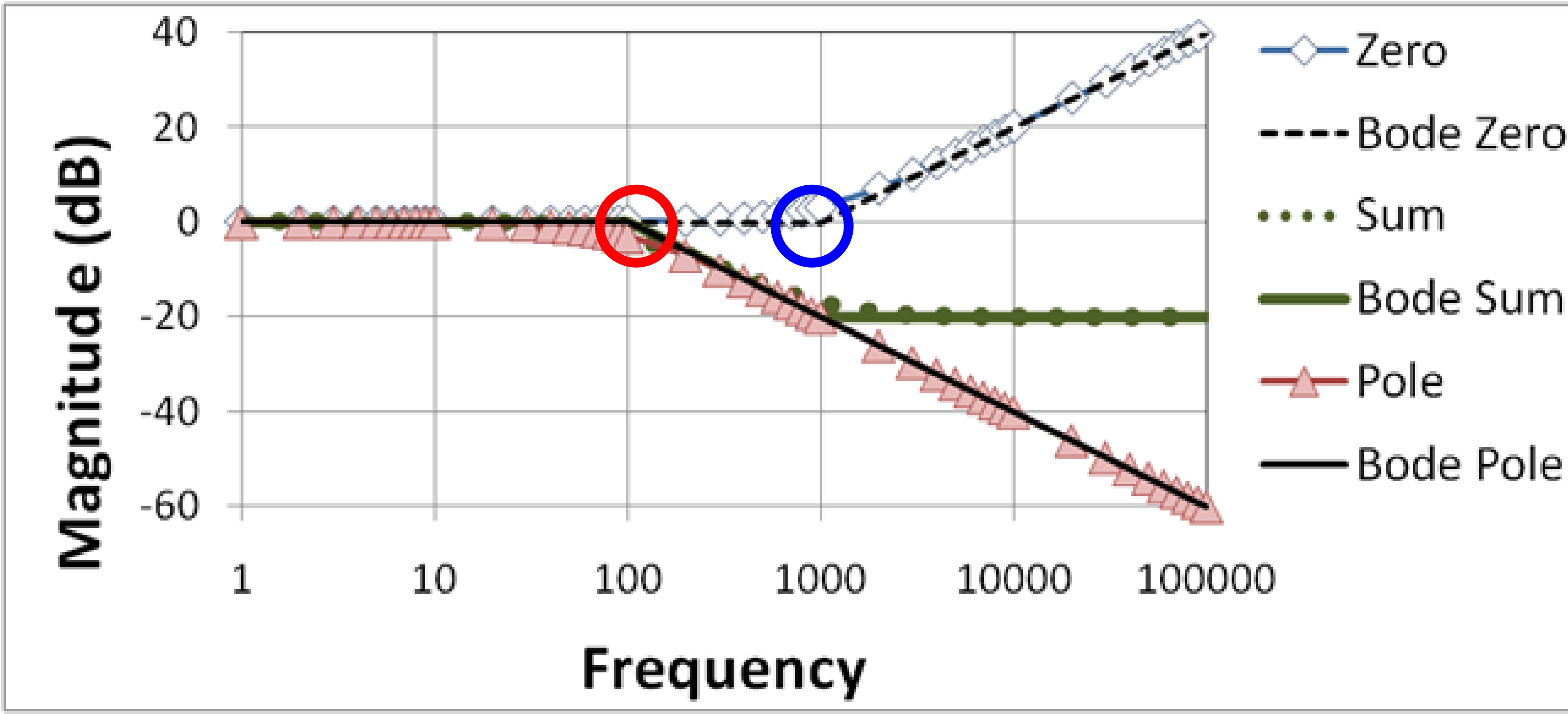
– Therefore plots for negative ω can be straightforwardly obtained from those of positive ω, so disregarded

– Log scale for frequency allows for covering a wider range of possible input frequencies on the same plot

# Bode Plot - Rules

- Zero (numerator = 0)
  - Amplitude: +20 dB/decade
  - Phase: +90º; +45º/decade, starting 1 decade before zero
- Pole (denominator = 0)
  - Amplitude: -20 dB/decade
  - Phase: -90º; -45º/decade, starting 1 decade before pole

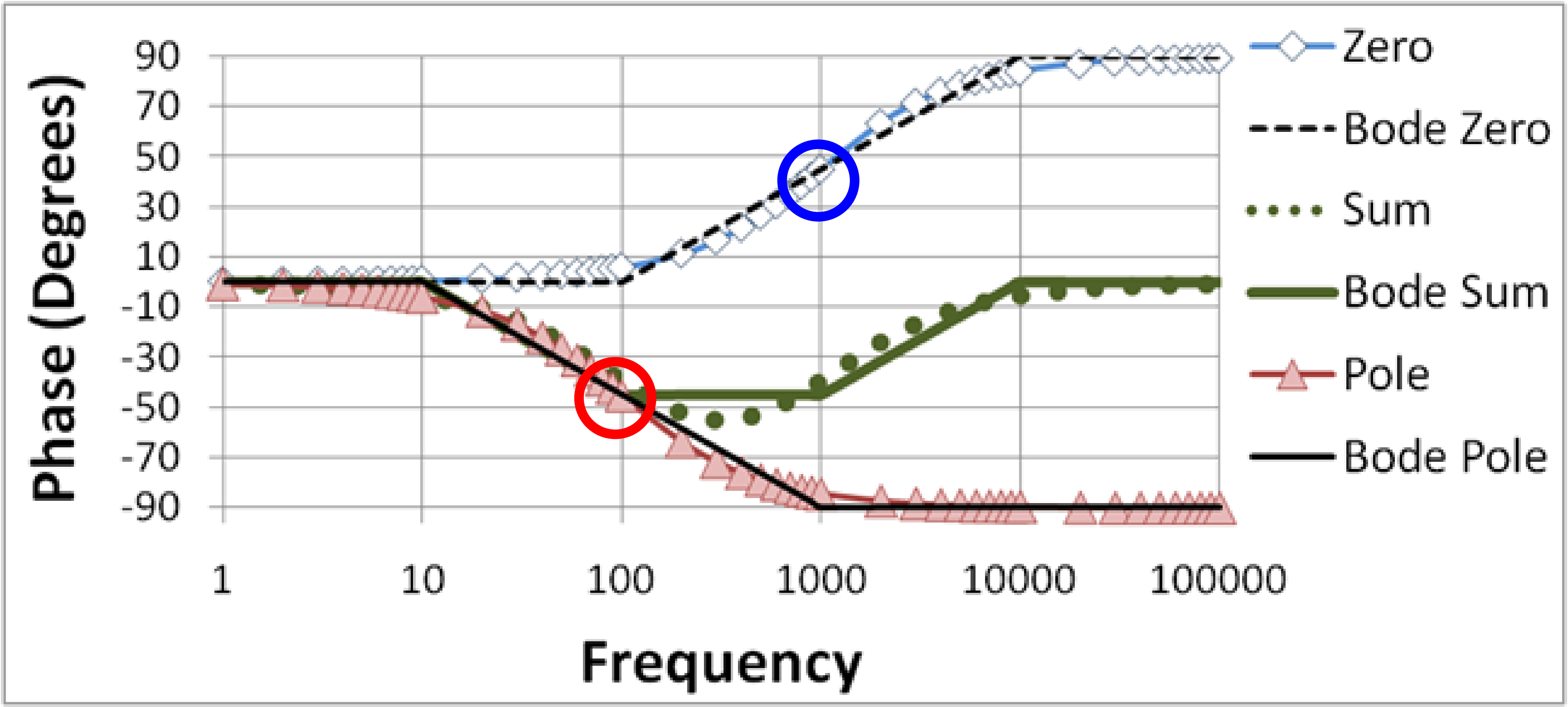
# Bode Plot (Magnitude)




**Zero (numerator = 0) Amplitude: +20 dB/decade**

**Pole (denominator = 0) Amplitude: -20 dB/decade**

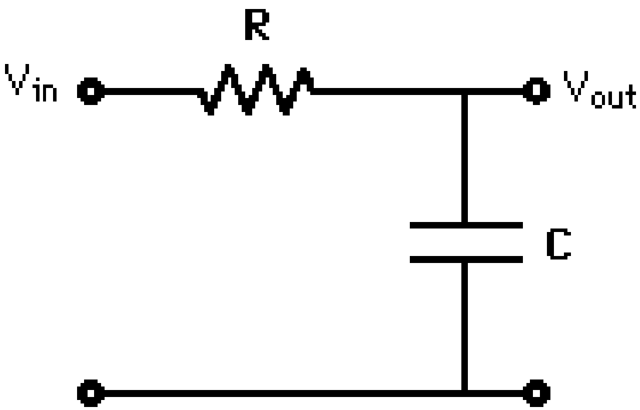
# Bode Plot (Phase)




**Zero (numerator = 0): +90º; 45º/decade, starting 1 decade before zero**

**Pole (denominator = 0): -90º; -45º/decade, starting 1 decade before pole**

# Example of Low-Pass Filter – From Transfer Function to Bode Plots




$$H(s) = \frac{1}{1 + sRC}$$

From s. 5:
1 pole s = -1/RC
i.e. $\omega = \left|-\frac{1}{RC}\right| = \frac{1}{RC}$

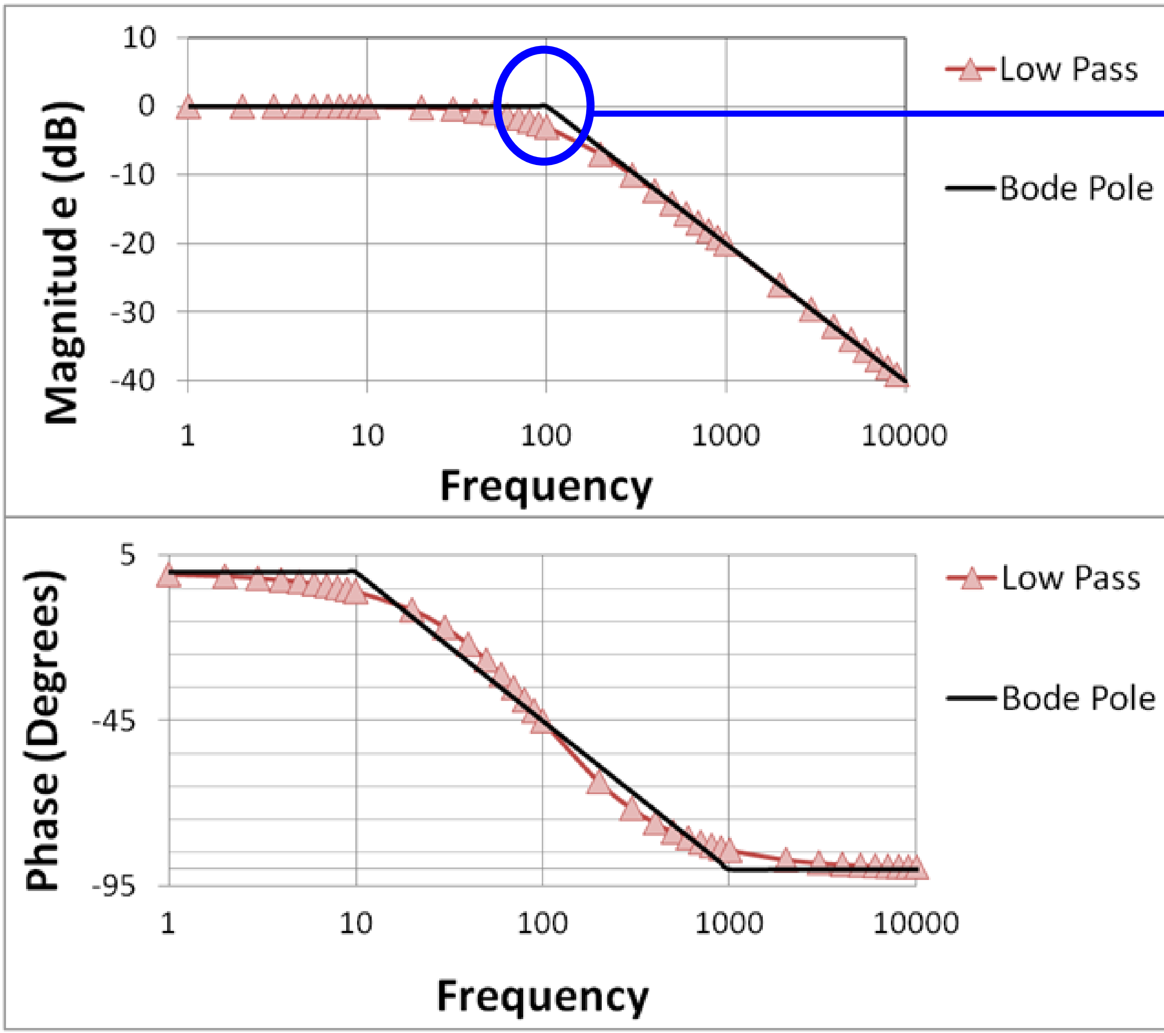
$$H(s = i\omega) = H(\omega) = \frac{1}{1 + i\omega RC}$$

Bode **magnitude**: $$|H(\omega)| = \left|\frac{1}{1 + i\omega RC}\right| = \frac{1}{|1 + i\omega RC|} = \frac{1}{\sqrt{1 + \omega^2 R^2 C^2}}$$

Bode **phase**: $$\measuredangle H(\omega) = \tan^{-1}\left(\frac{Im[H(\omega)]}{Re[H(\omega)]}\right) = -\tan^{-1}(\omega RC)$$

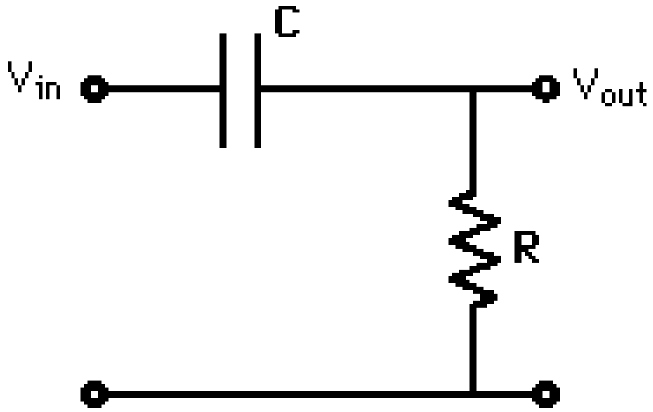
See also W4, s. 41

# Example: Low-Pass Filter – Bode Plot




See s. 42, W4: Break or cut-off frequency (-3 dB), $\omega = \frac{1}{RC} =$ 100 Hz

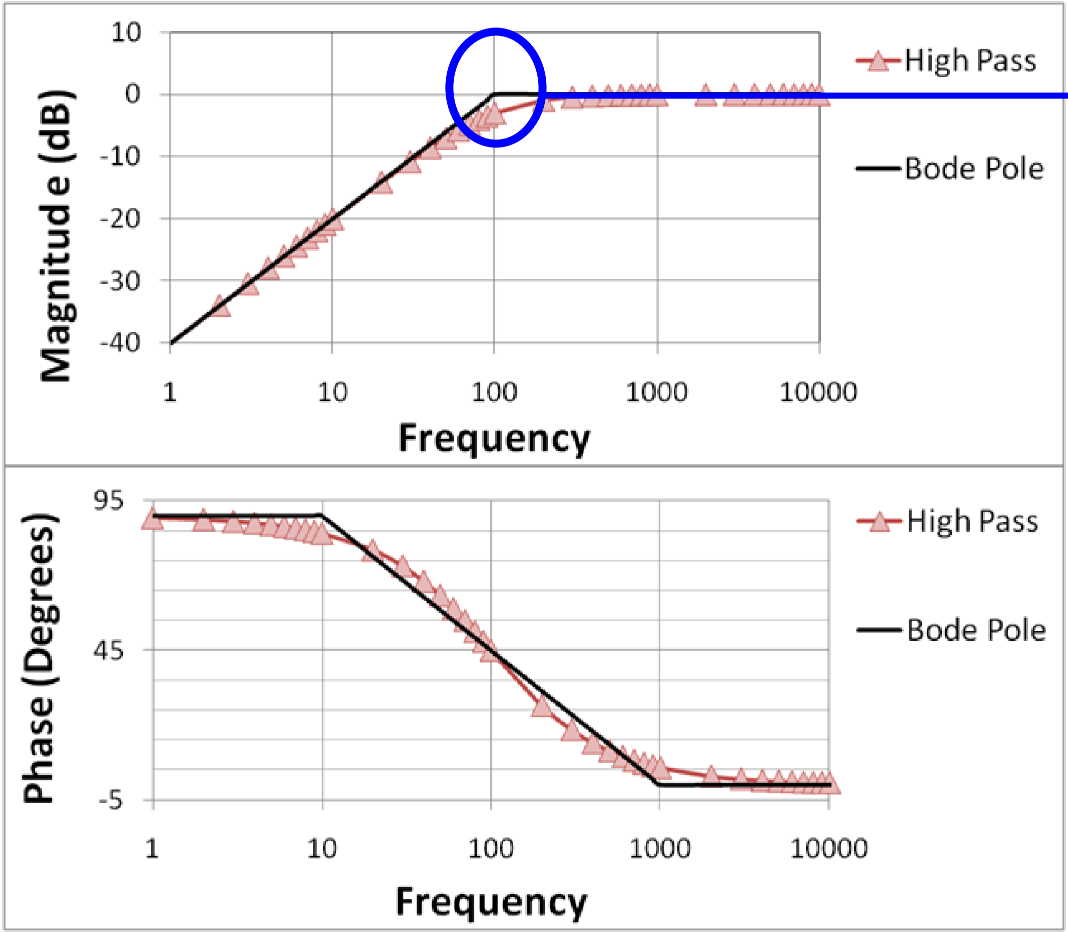
# Example: High-Pass Filter – Circuit




$$H(s) = \frac{sRC}{1 + sRC}$$

From s. 5:
1 zero at ω = 0
1 pole at ω = 1/RC

# Example: High-Pass Filter – Bode Plot




See s. 43, W4: Break or cut-off frequency (-3 dB), $\omega = \frac{1}{RC} =$ 100 Hz

# More Transforms for Time-Discrete Systems

# Discrete-Time Fourier Transform

- Corresponds to the Fourier Transform for discrete-time signals (different from the Discrete Fourier Transform, a finite, bounded approximation of the Fourier Transform for digital devices)
- Transform discrete-time signals from time-domain to frequency domain (continuous spectrum)

$$X(\omega) = \sum_{n=-\infty}^{\infty} x[n] \cdot e^{-i\omega n}$$

Note: compare with DFT notation on W2, s. 34:

$$X[k] = \sum_{n=0}^{N-1} x[n] \cdot e^{\frac{-2\pi i}{N} kn}$$

$$k = 0, \cdots, N-1$$

# Z-Transform

- Transform signals from time-domain to a complex frequency-domain representation (z-plane)
- Corresponds to Laplace transform for time-discrete signals
- The Discrete-Time Fourier Transform is a special case of the Z-Transform with $z = e^{i\omega}$

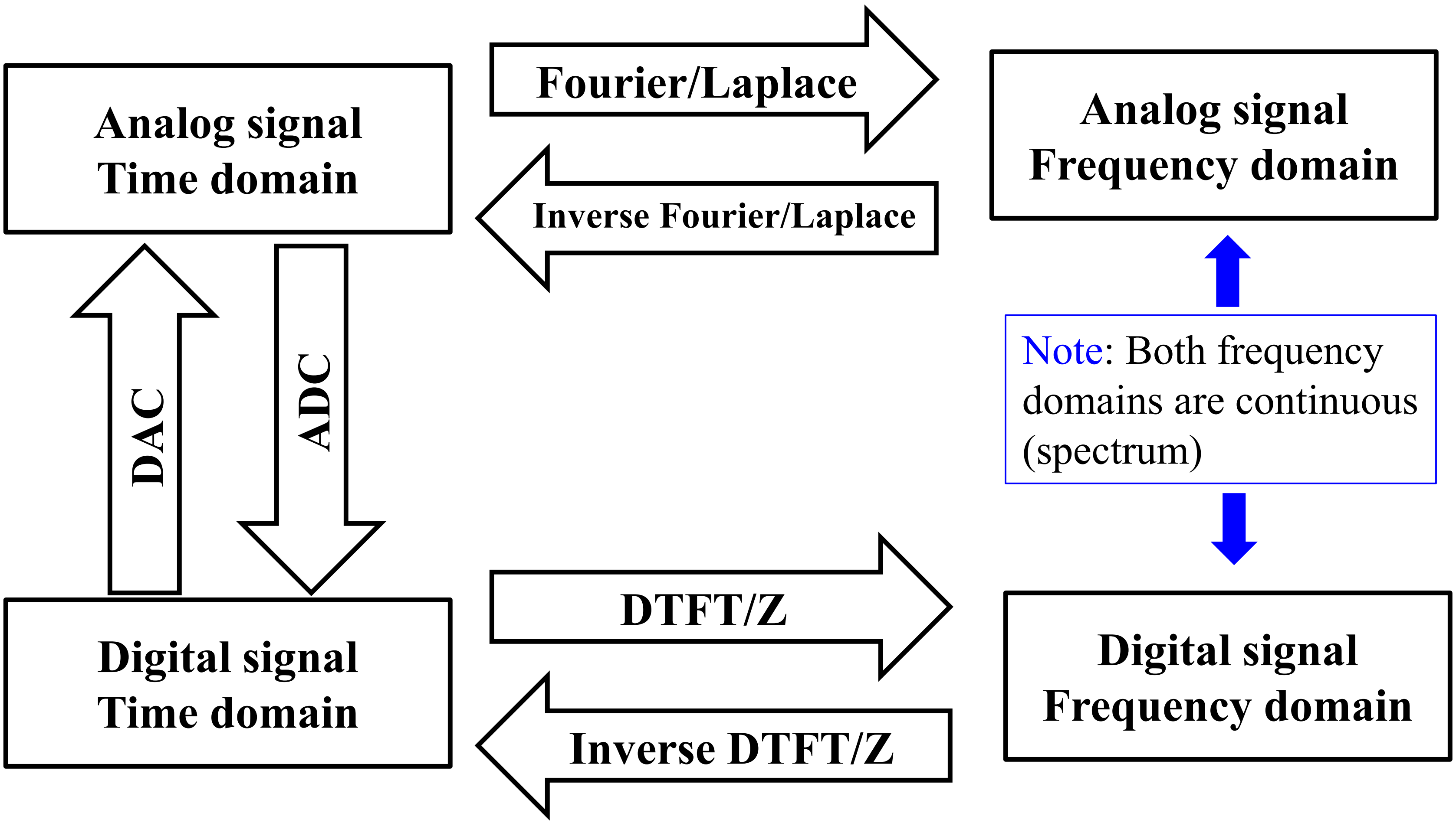
$$X(z) = Z\{x[n]\} = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

$$z = Ae^{i\phi} \text{ or } z = A(\cos\phi + i\sin\phi)$$

# Some Properties of Z-Transform

| Property | Signal | z-Transform |
|---|---|---|
| | $x[n]$ | $X(z)$ |
| | $x_1[n]$ | $X_1(z)$ |
| | $x_2[n]$ | $X_2(z)$ |
| Linearity | $ax_1[n] + bx_2[n]$ | $aX_1(z) + bX_2(z)$ |
| Time shifting | $x[n - n_0]$ | $z^{-n_0}X(z)$ |
| Scaling in the $z$-domain | $e^{j\omega_0 n}x[n]$ | $X(e^{-j\omega_0}z)$ |
| | $z_0^n x[n]$ | $X\left(\frac{z}{z_0}\right)$ |
| | $a^n x[n]$ | $X(a^{-1}z)$ |
| Time reversal | $x[-n]$ | $X(z^{-1})$ |
| Time expansion | $x_{(k)}[n] = \begin{cases} x[r], & n = rk \\ 0, & n \neq rk \end{cases}$ for some integer $r$ | $X(z^k)$ |
| Conjugation | $x^*[n]$ | $X^*(z^*)$ |
| Convolution | $x_1[n] * x_2[n]$ | $X_1(z)X_2(z)$ |
| First difference | $x[n] - x[n-1]$ | $(1 - z^{-1})X(z)$ |
| Accumulation | $\sum_{k=-\infty}^{n} x[k]$ | $\frac{1}{1 - z^{-1}}X(z)$ |
| Differentiation in the $z$-domain | $nx[n]$ | $-z\frac{dX(z)}{dz}$ |

# Transform Overview



# Frequency Responses and Transfer Functions for Time-Discrete LTI Systems

# Frequency Response

From W4, s. 26

**Continuous time**

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{ik\omega_0 t} \longrightarrow \boxed{h(t)} \longrightarrow y(t) = \sum_{k=-\infty}^{\infty} H(k\omega_0) a_k e^{ik\omega_0 t}$$

$$a_k \to \underbrace{H(k\omega_0)}_{\textbf{Gain}} a_k$$

$$H(k\omega_0) = \underbrace{|H(k\omega_0)|}_{\textbf{Amplitude}} e^{\overbrace{i\measuredangle H(k\omega_0)}^{\textbf{Phase}}}$$

By linearity a sum of frequencies go out of the LTI filter only with different amplitude and phase.

# Frequency Response

## Discrete time

$$x[n] = \sum_{k=-\infty}^{\infty} a_k e^{ik\omega_0 n} \longrightarrow \boxed{h[n]} \longrightarrow y[n] = \sum_{k=-\infty}^{\infty} H(k\omega_0) a_k e^{ik\omega_0 n}$$

$$a_k \to \underbrace{H(k\omega_0)}_{\textbf{Gain}} a_k$$

$$H(k\omega_0) = \underbrace{|H(k\omega_0)|}_{\textbf{Amplitude}} e^{\overbrace{i\measuredangle H(k\omega_0)}^{\textbf{Phase}}}$$

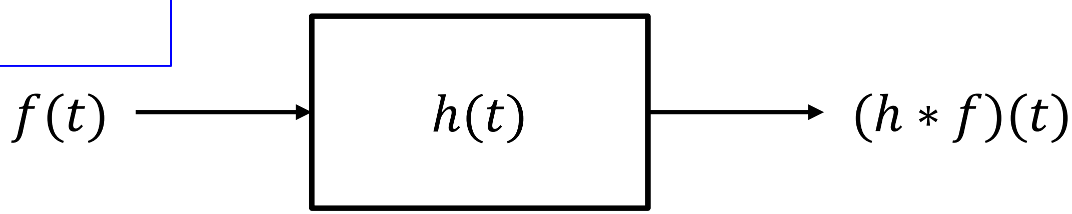
By linearity a sum of frequencies go out of the LTI filter
only with different amplitude and phase.

# Time-Continuous Transfer Function

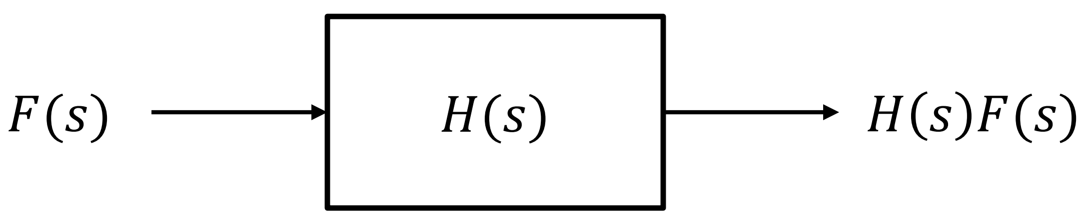
h(t) : impulse response (CT)
H(s) : transfer function (stationary and transient frequency response) of the filter/system, i.e Laplace transform of h(t)

From W4, s. 27

**Time domain**




**Complex frequency domain (s-plane)**

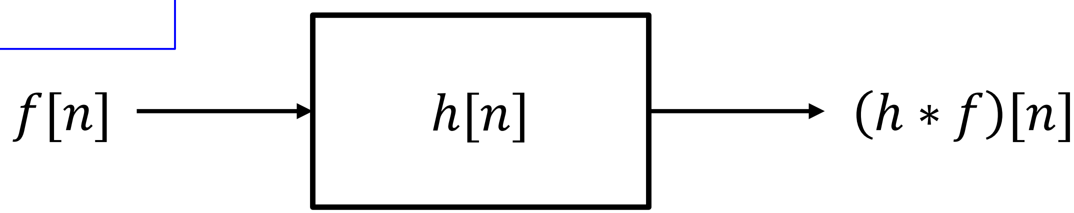



Reminder: a convolution in the time domain is a multiplication in the frequency domain

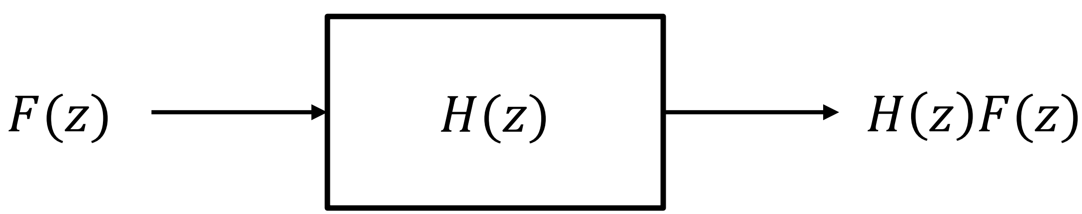
# Time-Discrete Transfer Function

h[n] : impulse response (DT)
H(z) : transfer function (stationary and transient frequency response) of the filter/system, i.e. Z-transform of h[n]

**Time domain**




**Complex frequency domain (z-plane)**




Reminder: a convolution in the time domain is a multiplication in the frequency domain

# From Transfer Functions to Frequency Reponses

- Typical configuration when digital filtering is applied to analog signals:


- Frequency response useful when input x[n] = sum of sinusoids, then output y[n] also sum of sinusoids
- Focus on the spectrum

# From Transfer Functions to Frequency Reponses

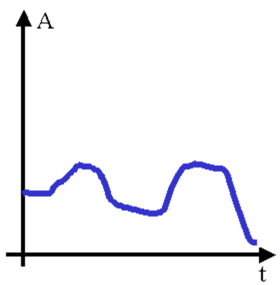
- Bode plots have been defined for continuous-time systems (e.g., analog filters)
- With z = $e^{i\omega}$ (see s. 18), the Z-transform degrades to a DTFT (as the Laplace transform was degrading to the FT with s = $i\omega$)
- We can therefore calculate the frequency response of the corresponding discrete-time system (e.g., a digital filter) with the transfer function $H(e^{i\omega})$
- A FFT is then typically used to calculate numerically the frequency response on computers
- Matlab has a dedicated function for this: freqz

Note: response in normalized angular frequency

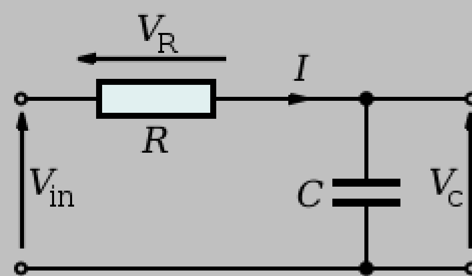
# Digital Filters

# Filters as System Examples

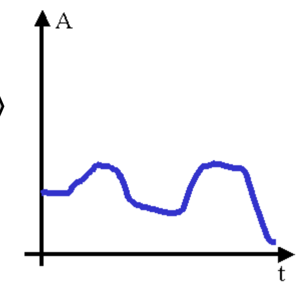
Analog



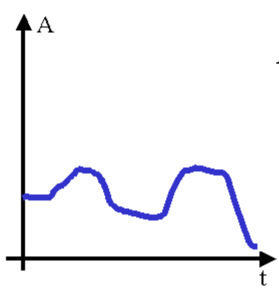

Circuit




Analog filter

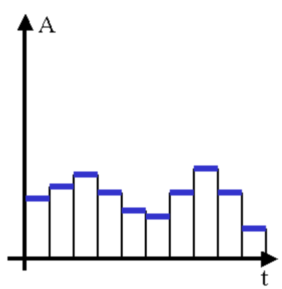



Analog




**A/D**

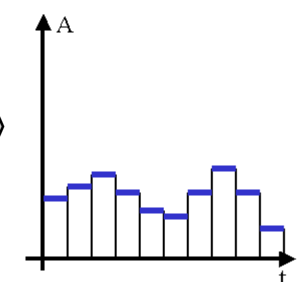
Digital




Function/algorithm

$$y_1 \ldots y_n = f(x_1 \ldots x_n)$$

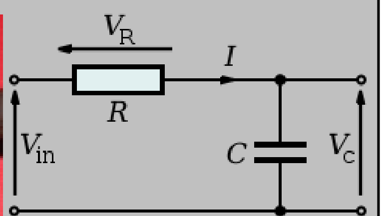
Digital filter



# Transfer Functions of Filters

## Analog

**Circuit**




**Laplace Transf.** →

$$H(s) = \frac{v_c}{v_{in}} = \frac{\overbrace{1}^{\text{Numerator}}}{\underbrace{1+RCs}_{\text{Denominator}}}$$

## Digital

**Function/algorithm**

$$y_1 \ldots y_n = f(x_1 \ldots x_n)$$

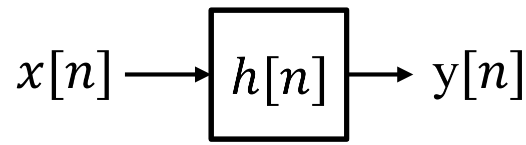
**Z Transf.** →

$$H(z) = \frac{\overbrace{b_0 + b_1 z^{-1} + \ldots + b_N z^{-N}}^{\text{Numerator}}}{\underbrace{1 + a_1 z^{-1} + \ldots + b_M z^{-M}}_{\text{Denominator}}}$$

# General Representation (Causal Filters)

**Difference equation:**

$$y[n] = -\sum_{k=1}^{N} a_k y[n-k] + \sum_{k=0}^{M} b_k x[n-k]$$



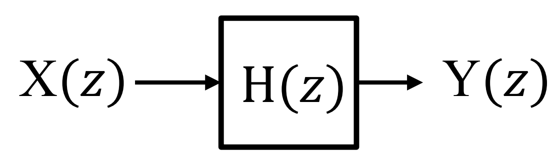

**Z-transform:**

$$Y(z) + \sum_{k=1}^{N} a_k Y(z) z^{-k} = \sum_{k=0}^{M} b_k X(z) z^{-k}$$

From s. 18, Z-transform properties:
$Z\{x[n-n_0]\} = z^{-n_0} \mathrm{X}(\mathrm{z})$

**Transfer function:**

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 + \sum_{k=1}^{N} a_k z^{-k}}$$



# An Example

**Difference equation:**

$$y[n] = x[n] + 2x[n-1] + x[n-2] - \frac{1}{4}y[n-1] + \frac{3}{8}y[n-2]$$

**Z-transform:**

$$Y(z) + \frac{1}{4}Y(z)z^{-1} - \frac{3}{8}Y(z)z^{-2} = X(z) + 2X(z)z^{-1} + X(z)z^{-2}$$

**Transfer function:**

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1 + 2z^{-1} + z^{-2}}{1 + \frac{1}{4}z^{-1} - \frac{3}{8}z^{-2}} = \frac{z^2 + 2z + 1}{z^2 + \frac{1}{4}z - \frac{3}{8}} = \frac{(z+1)^2}{\left(z - \frac{1}{2}\right)\left(z + \frac{3}{4}\right)}$$

Note: poles and zeros of H(z) can be represented in the zero-pole plot (complex z-plane) for further analysis (e.g., stability).

# FIR Filters

# Nonrecursive Digital Filters

- $a_k = 0$ for all k
- $y[n] = \sum_{k=0}^{M} b_k\, x[n-k]$
- Finite Impulse Response (FIR)
- The FIR filter above is a causal system: the output depends only on past values of inputs
- The filter coefficients $b_k$ define the FIR filter
- The filter order is M, the number of coefficients $b_k$ is M+1 (filter "length")

# Examples of FIR Filters

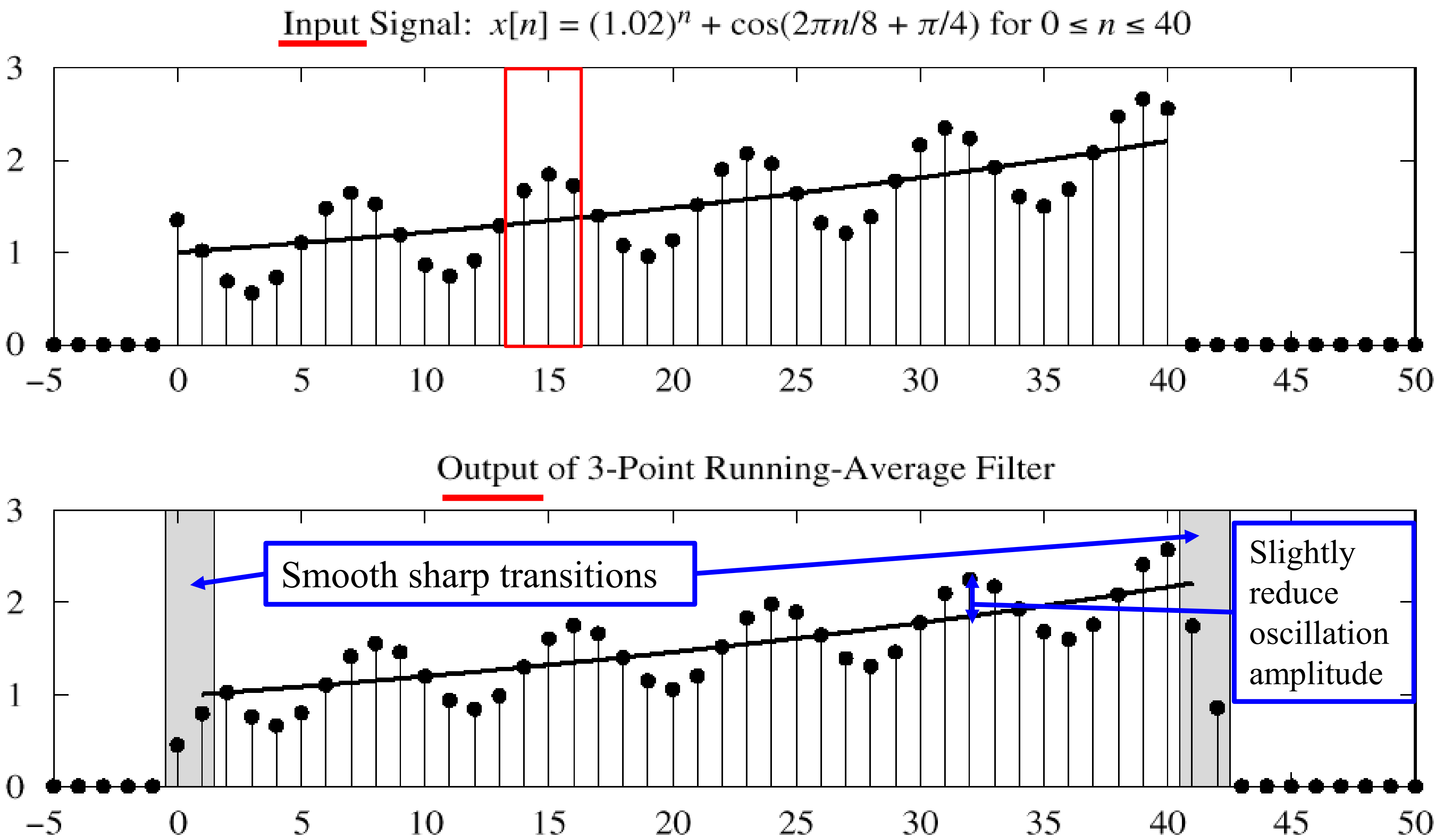
- 3-point moving average:

$$y_3[n] = \sum_{k=0}^{2} \frac{1}{3} x[n-k] = \frac{1}{3} x[n] + \frac{1}{3} x[n-1] + \frac{1}{3} x[n-2]$$

- 7-point moving average:

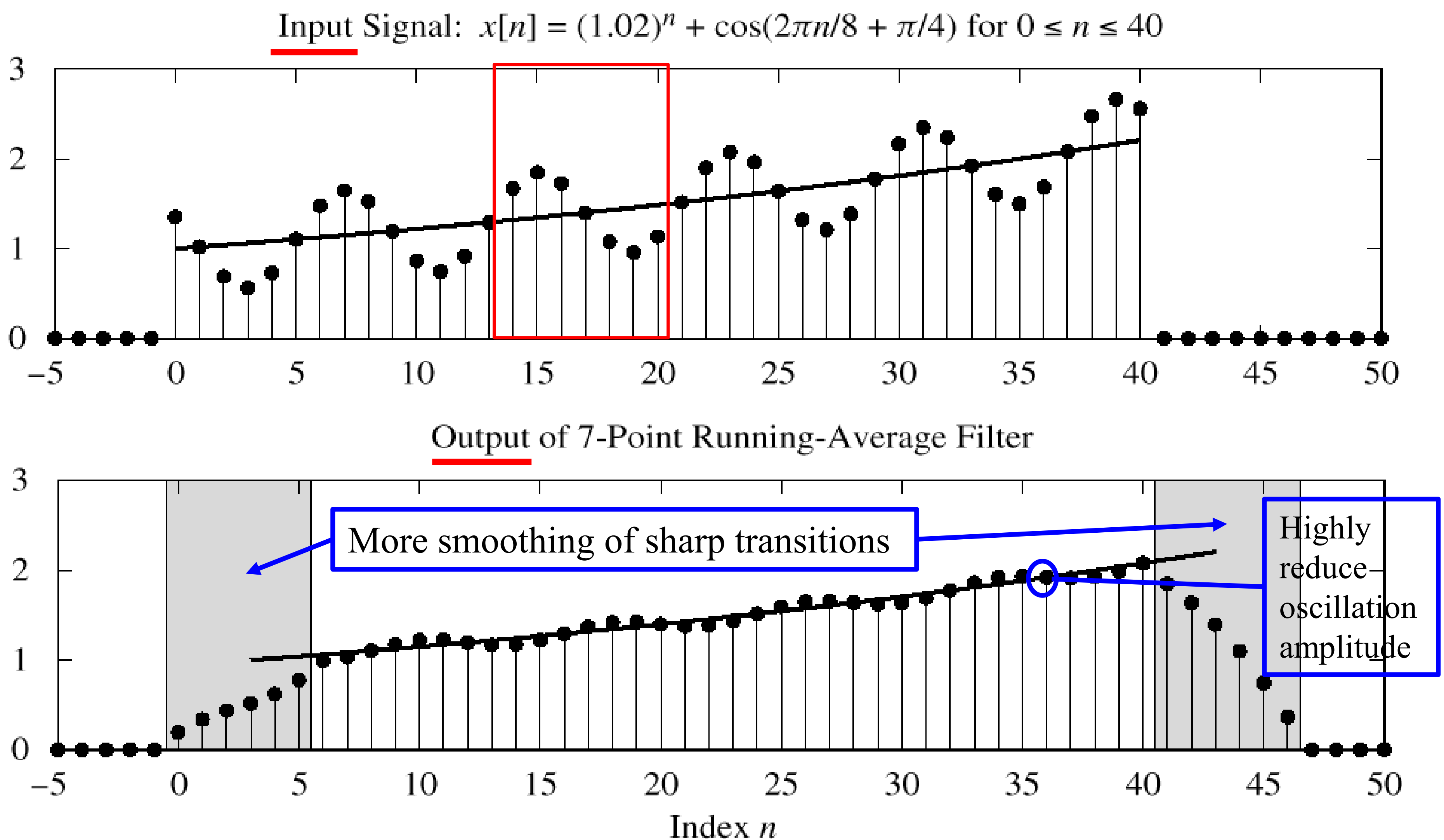
$$y_7[n] = \sum_{k=0}^{6} \frac{1}{7} x[n-k] = \frac{1}{7} x[n] + \frac{1}{7} x[n-1] + \cdots + \frac{1}{7} x[n-6]$$

# 3-pt Moving Average Example



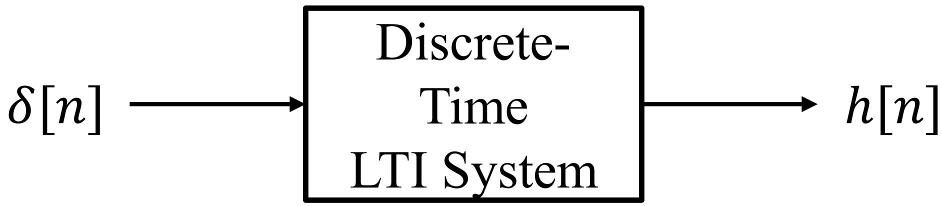

[Picture from McClellan, Schafer, and Yoder, "DSP First: A Multimedia Approach"]

# 7-pt Moving Average Example

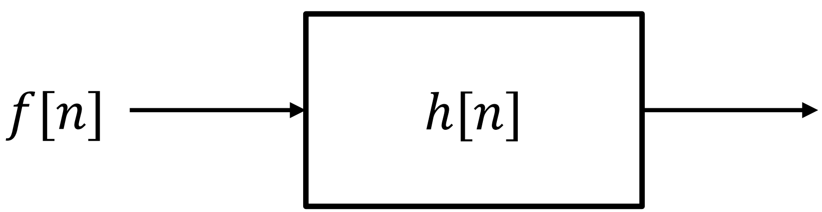



[Picture from McClellan, Schafer, and Yoder, "DSP First: A Multimedia Approach"]

# Impulse Response of a Discrete-Time LTI System




From W4, s. 14-15

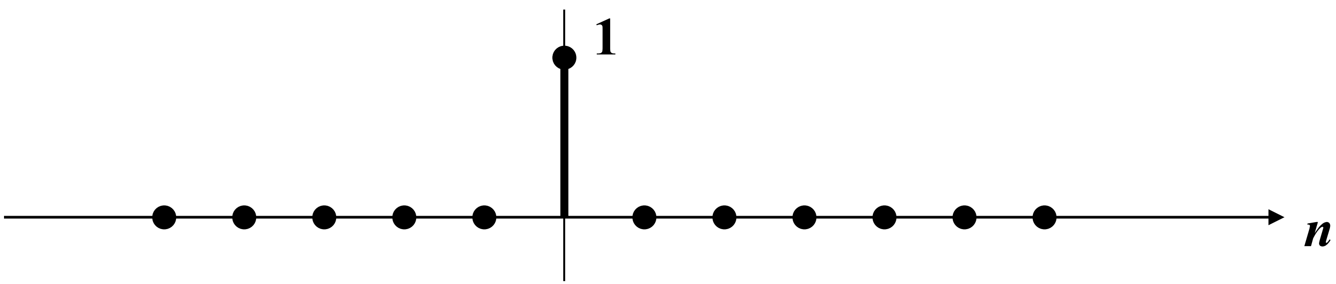



$$(f * h)[n] = \sum_{m=-\infty}^{\infty} f[m]h[n-m]$$

# The Unit Impulse

$$\delta[n] = \begin{cases} 1 & n = 0 \\ 0 & n \neq 0 \end{cases}$$

Notation:
Kronecker delta function

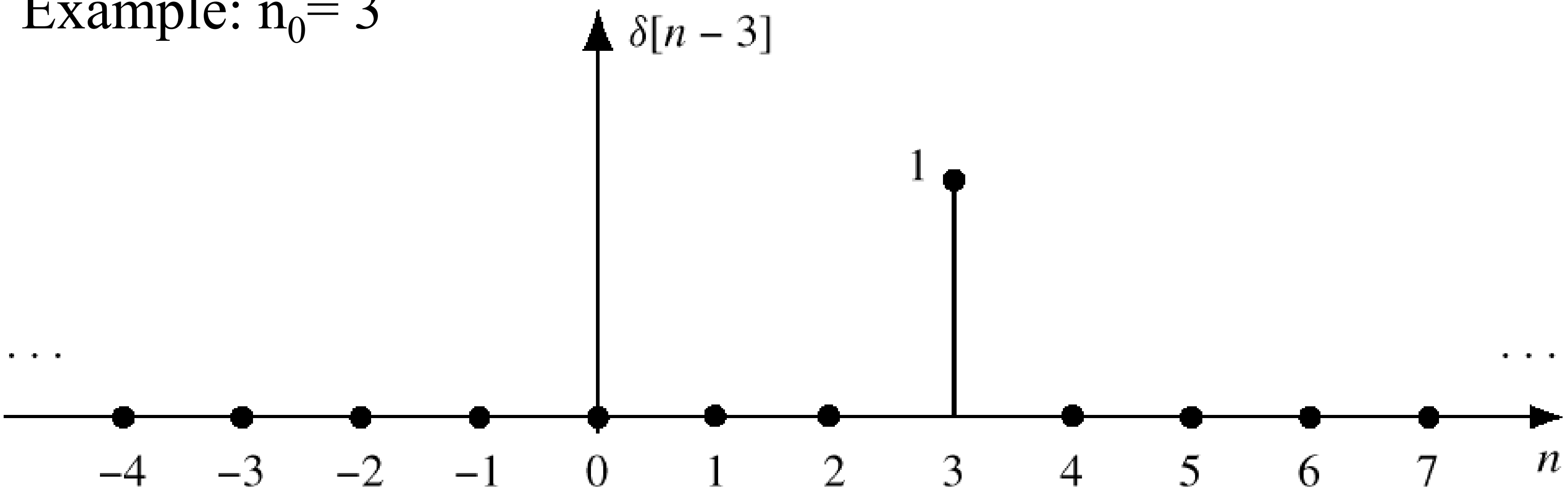



[Picture from McClellan, Schafer, and Yoder, “DSP First: A Multimedia Approach”]

# Time-Shifted Unit Impulse

$$\delta[n-n_0] = \begin{cases} 1 & n = n_0 \\ 0 & n \neq n_0 \end{cases}$$

Example: $n_0 = 3$




[Picture from McClellan, Schafer, and Yoder, "DSP First: A Multimedia Approach"]

# Impulse Response of a FIR Filter

$$y[n] = \sum_{k=0}^{M} b_k x[n-k]$$

A FIR filter usually described in terms of coefficients $b_k$

Alternatively, as any other LTI system, we can describe a FIR filter using its impulse response.

If $x[n] = \delta[n]$ then $\mathrm{y}[n] = h[n]$, i.e. the impulse response

$$\Longrightarrow \quad \boxed{h[n] = \sum_{k=0}^{M} b_k \delta[n-k]}$$

Note: you can check $(x * h)[n] = \mathrm{y[n]}$

# Impulse Response of a FIR Filter

$$h[n] = \sum_{k=0}^{M} b_k \delta[n-k]$$

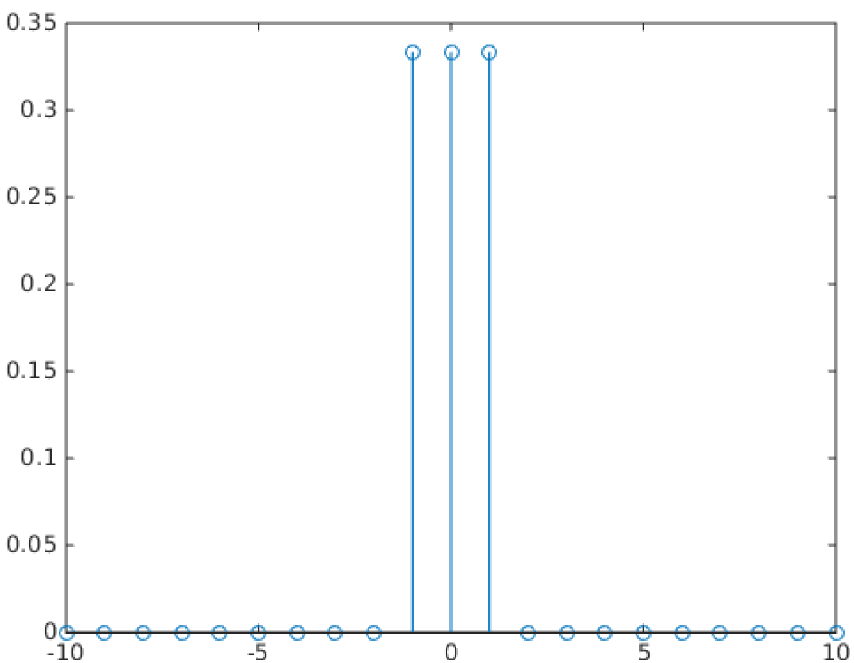
| $n$ | $n < 0$ | 0 | 1 | 2 | 3 | $\dots$ | $M$ | $M+1$ | $n > M+1$ |
|---|---|---|---|---|---|---|---|---|---|
| $x[n] = \delta[n]$ | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $y[n] = h[n]$ | 0 | $b_0$ | $b_1$ | $b_2$ | $b_3$ | $\dots$ | $b_M$ | 0 | 0 |

Impulse response is finite!

[Table from McClellan, Schafer, and Yoder, "DSP First: A Multimedia Approach"]

# Example of FIR Filter: Noncausal 3-Point Moving Average

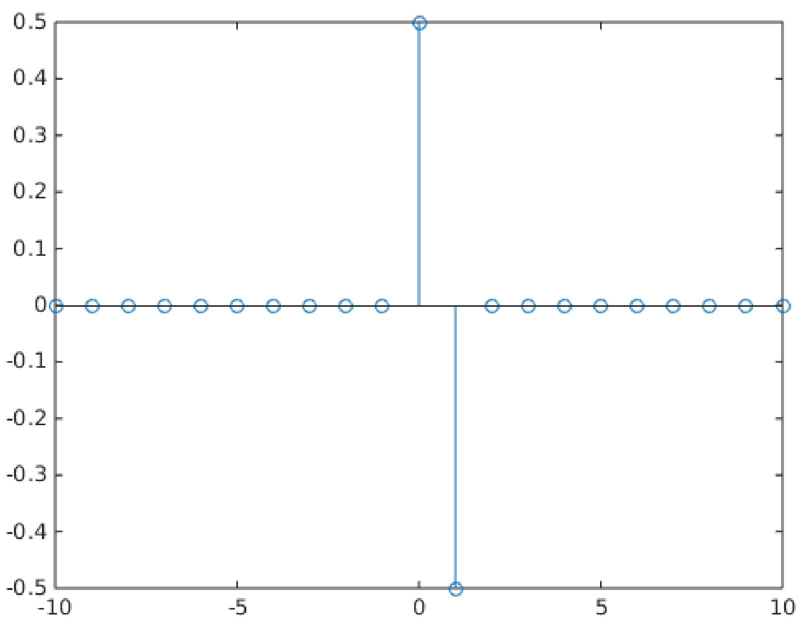
- $b_k = \{\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\}$ for k= -1,0,1 **Filter coefficients**
- $y[n] = \frac{1}{3}(x[n-1] + x[n] + x[n+1])$ **Difference equation**
- $h[n] = \frac{1}{3}(\delta[n-1] + \delta[n] + \delta[n+1])$ **Impulse response**




[Picture from Prof. A. S. Willsky, Signals and Systems course]

# Example of FIR Filter: Causal High-Pass Filter

- $b_k = \{\frac{1}{2}, -\frac{1}{2}\}$ for k= 0,1 **Filter coefficients**
- $y[n] = \frac{1}{2}(x[n] - x[n-1])$ **Difference equation**
- $h[n] = \frac{1}{2}(\delta[n] - \delta[n-1])$ **Impulse response**




[Picture from Prof. A. S. Willsky, Signals and Systems course]

# IIR Filters

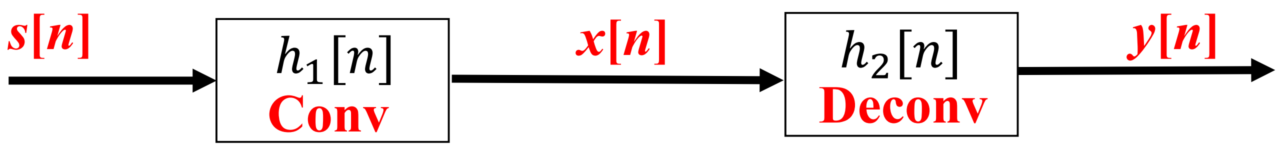
# Motivation


Can we remove the blur in postprocessing?
Yes! With a deconvolution filter!

[Picture from McClellan, Schafer, and Yoder, "DSP First: A Multimedia Approach"]

# Motivation

Original Image | Blurred (Motion) | Restored w/ Inverse Filter




Given $h_1[n]$, can we find $h_2[n]$ so that $y[n] = s[n]$?

$$x[n] = s[n] * h_1[n]$$
$$y[n] = x[n] * h_2[n] = s[n] * h_1[n] * h_2[n]$$
$$\Rightarrow h_1[n] * h_2[n] = \delta[n] \text{ (convolution with unit impulse} \leftrightarrow \text{identity)}$$

[Adapted from McClellan, Schafer, and Yoder, “DSP First: A Multimedia Approach”]

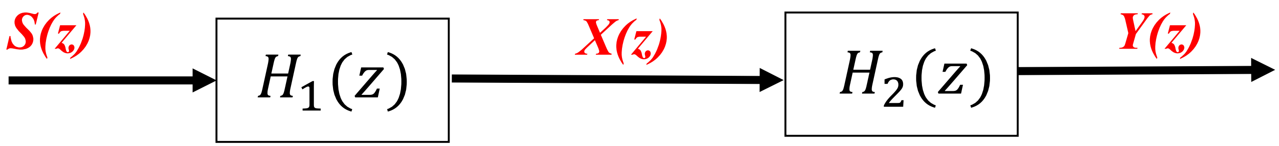
# Motivation

Blurring filter with parameter a:

$$x[n] = s[n] - as[n-1] \Rightarrow h_1[n] = \delta[n] - a\delta[n-1]$$

Difficult to solve in time domain within convolution sum.

Z-domain:




$$Y(z) = H_2(z)H_1(z)S(z) = H(z)S(z)$$
$$H(z) = 1 = H_2(z)H_1(z)$$
$$\Rightarrow H_2(z) = 1/H_1(z)$$

Blurring filter in z-domain (from DE above): $H_1(z) = 1 - az^{-1}$

$$\Rightarrow H_2(z) = \frac{1}{1 - az^{-1}}$$

[Adapted from McClellan, Schafer, and Yoder, "DSP First: A Multimedia Approach"]

# Motivation

$$H_2(z) = \frac{1}{1 - az^{-1}}$$

What type of filter?

$$H_2(z) = \frac{Y(z)}{X(z)} = \frac{1}{1 - az^{-1}}$$

$$Y(z) = aY(z)z^{-1} + X(z)$$

$$y[n] = ay[n-1] + x[n]$$

If a $\neq 0$ not a FIR (see s. 34)!

In fact $H_2$ is a first order IIR filter!

# Recursive Digital Filters

- $y[n] = -\sum_{k=1}^{N} a_k\, y[n-k] + \sum_{k=0}^{M} b_k\, x[n-k]$
- $a_k \neq 0$ and $b_k \neq 0$ at least for one k
- $a_k$: feedback coefficients
- $b_k$: feed-forward coefficients
- Infinite Impulse Response (IIR)
- The IIR filter above is a causal system: the output depends only on past values of output and input
- The filter order is typically N and the total number of coefficients is N+M+1

# Impulse Response of a IIR Filter

Example: generalized first order IIR filter:

$$y[n] = a_1 y[n-1] + b_0 x[n] \Rightarrow h[n] = a_1 h[n-1] + b_0 \delta[n]$$

| $n$ | $n<0$ | 0 | 1 | 2 | 3 | 4 | $\cdots$ |
|---|---|---|---|---|---|---|---|
| $\delta[n]$ | 0 | 1 | 0 | 0 | 0 | 0 | $\cdots$ |
| $h[n-1]$ | 0 | 0 | $b_0$ | $b_0(a_1)$ | $b_0(a_1)^2$ | $b_0(a_1)^3$ | $\cdots$ |
| $h[n]$ | 0 | $b_0$ | $b_0(a_1)$ | $b_0(a_1)^2$ | $b_0(a_1)^3$ | $b_0(a_1)^4$ | $\cdots$ |

Impulse response is infinite!

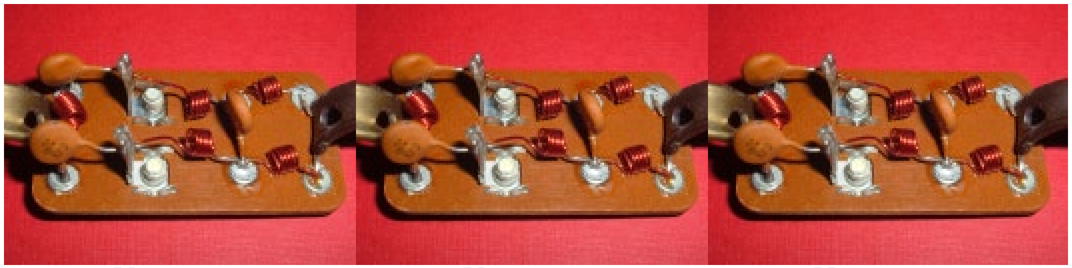
# Order and Types of Filters

# Filter Order and Type

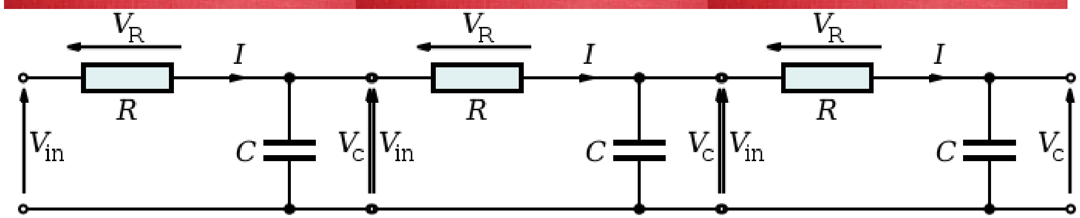
- Several filters exist in both analog and digital form and are defined by the polynomials at the numerator/denominator (Bessel, Butterworth, Tschebishev, etc.)
- 1st order is equivalent to 20dB per decade
- Each successive order adds 20dB per decade
- Filter with a high order are closer to the ideal filter (rectangular function)

# Filter Order

## Analog



…




…

Filter order: 3

++ faster cutoff

-- more components

-- higher power consumption

## Digital

$$y[n] = b_0 x[n]$$ Filter order: 0

$$y[n] = b_0 x[n] + b_1 x[n-1]$$ Filter order: 1

$$y[n] = b_0 x[n] + b_1 x[n-1] + b_2 x[n-2]$$ Filter order: 2

…

++ faster cutoff

-- more computation

-- higher power consumption

# Conclusion

# Take Home Messages

- Bode plots reproduce an estimated frequency response of (cascaded) LTI; they have been developed for continuous-time systems and expanded to time-discrete systems
- The Z-transform is the dual of the Laplace transform for time-discrete systems
- Both analog and digital filters are:
  - characterized by different order and coefficient distributions
  - they can be expressed in time and (complex) frequency domains
- Several equivalent forms to define digital filters are possible:
  - time domain: coefficient set, difference equation, impulse response
  - frequency domain: transfer function
- The impulse response of a digital filter which can be finite (filter non recursive) or infinite (filter recursive); FIR and IIR filters are correspondingly defined

# Additional Literature – Week 5

Books

- J. H. McClellan, R. W. Schafer, M. A. Yoder "DSP First: A Multimedia Approach", Prentice Hall, 1999.
- A. Oppenheim and A. S. Willsky with S. Nawab, "Signals and Systems", Prentice Hall, 1997.