Signals, Instruments, and Systems – W10

An Introduction to Mobile Robotics – Localization in Presence of Uncertainties
Outline

- Error sources in odometry
  - Deterministic
  - Non-deterministic

- Odometry in 1D and 2D
  - Based on accelerometers
  - Based on wheel-encoders/step counters

- Feature-based localization

- Fusing information from noisy processes and noisy sensing: the Kalman Filter algorithm in 1D problems
Deterministic Error Sources in Wheel-Based Odometry
Deterministic Error Sources

- Limited encoder resolution → Intrinsic max accuracy
- Wheel misalignment, small differences in wheel diameter and inter-wheel axis → Can be fixed by calibration
Pose Variation During $\Delta t$

- Influenced by encoder/stepper counter resolution (in blue)
- Parameters to be possibly calibrated (in red)

\[
p = \begin{bmatrix} x \\ y \\ \theta \end{bmatrix} \quad t' = t + \Delta t \quad p' = \begin{bmatrix} x' \\ y' \\ \theta' \end{bmatrix}
\]

\[
\Delta s_r = v_r \Delta t = \Delta \phi_r r_r
\]

\[
\Delta s_l = v_l \Delta t = \Delta \phi_l r_l
\]

\[
p' = f(x, y, \theta, \Delta s_r, \Delta s_l, b) = \begin{bmatrix} x \\ y \\ \theta \end{bmatrix} + \begin{bmatrix} \Delta s \cos(\theta + \Delta \theta/2) \\ \Delta s \sin(\theta + \Delta \theta/2) \\ \Delta \theta \end{bmatrix} = \begin{bmatrix} x \\ y \\ \theta \end{bmatrix} + \begin{bmatrix} \frac{\Delta s_r + \Delta s_l}{2} \cos(\theta + \frac{\Delta s_r - \Delta s_l}{2b}) \\ \frac{\Delta s_r + \Delta s_l}{2} \sin(\theta + \frac{\Delta s_r - \Delta s_l}{2b}) \\ \frac{\Delta s_r - \Delta s_l}{b} \end{bmatrix}
\]

\[r_r = \text{right wheel radius}\]
\[r_l = \text{left wheel radius}\]
\[b = \text{inter-wheel distance}\]
\[\Delta s_r = \text{traveled distance right wheel}\]
\[\Delta s_l = \text{traveled distance left wheel}\]
\[\Delta \theta = \text{orientation change of the vehicle}\]

See also Week 9, s. 33-35

• Influenced by encoder/stepper counter resolution (in blue)
• Parameters to be possibly calibrated (in red)
Non-Deterministic Error Sources in 1D Localization
Non-Deterministic Error Sources

• From Week 5: no deterministic prediction possible → we have to describe them **probabilistically**

• Example: accelerometer-based odometry

MEMS-Based accelerometer (e.g., on DISAL Arduino kit or e-puck)
Odometry in 1D

\[ X_0 = 0 \]

\[ t = 0 \]
Odometry in 1D

\[
\begin{align*}
\hat{X}_1 &= X_0 + (a_1 + \ddot{a}_1) \Delta t^2 \\
X_1 &= X_0 + a_1 \Delta t^2
\end{align*}
\]
Odometry in 1D

\[ \hat{X}_2 = \hat{X}_1 + (a_2 + \ddot{a}_2) \Delta t^2 \]

\[ X_2 = X_1 + a_2 \Delta t^2 \]
Odometry in 1D

\[ \hat{X}_3 = \hat{X}_2 + (a_3 + \ddot{a}_3) \Delta t^2 \]

\[ X_3 = X_2 + a_3 \Delta t^2 \]
Odometry in 1D

\[ \hat{X}_4 = \hat{X}_3 + (a_4 + \ddot{a}_4) \Delta t^2 \]

\[ X_4 = X_3 + a_4 \Delta t^2 \]
Odometry in 1D

\[ \hat{X}_5 = \hat{X}_4 + (a_5 + \ddot{a}_5) \Delta t^2 \]

\[ X_5 = X_4 + a_5 \Delta t^2 \]
1D Odometry: Error Modeling

- Error happens!
- Odometry error is cumulative.
  → grows without bound
- We need to be aware of it.
  → We need to model odometry error.
  → We need to model sensor error.
- Multiple independent source of errors with arbitrary distribution combined → Central Limit Theorem → Gaussian assumption reasonable
- Acceleration $a$ is a random variable drawn from “mean-free” Gaussian (“Normal”) distribution.
  → Position $X$ is random variable with Gaussian distribution.
1D Odometry with Gaussian Uncertainty

\[ t=0 \]

Estimated position

True position

$X[m]$
1D Odometry with Gaussian Uncertainty

\[ \mu = 1 \]
\[ \sigma = 0.2 \]
1D Odometry with Gaussian Uncertainty

\[ t = 2 \]

\[ \mu = 2 \]

\[ \sigma = 0.4 \]
1D Odometry with Gaussian Uncertainty

At $t=3$, the estimated position is $\mu = 3$ with a standard deviation $\sigma = 0.6$. The true position is indicated by the red crosses.
1D Odometry with Gaussian Uncertainty

\[ t=4 \]

\[ \mu = 4 \]

\[ \sigma = 0.8 \]
1D Odometry with Gaussian Uncertainty

\[ t = 5 \]

\[ \mu = 5 \]

\[ \sigma = 1 \]
Features

- Odometry based position error grows without bound.
- Use relative measurement to features ("landmarks") to reduce position uncertainty

**Feature:**
- Uniquely identifiable
- Position is known
- We can obtain relative measurements between robot and feature (usually angle or range).

**Examples:**
- Doors, walls, corners, hand rails
- Buildings, trees, lanes
- GNSS satellites
Automatic Feature Extraction

- High-level features:
  - Doors, persons
- Simple visual features:
  - Edges (Canny Edge Detector 1983)
  - Corner (Harris Corner Detector 1988)
- Simple geometric features
  - Lines
  - Corners
- “Binary” feature

Complexity
Feature-Based Localization

\[ t = 5 \]

\[ \mu = 5; \sigma = 1 \]
Feature-Based Localization

$t = 5$

$\mu = 5, \sigma = 1$
Feature-Based Localization

$t=5$

$\mu = 5; \sigma = 1$

$r = 3.2m$
Feature-Based Localization

\[ t = 5 \]

\[ \mu = 5; \sigma = 1 \]

\[ \mu = 5.8; \sigma = 1.2 \]

\[ r = 3.2m \]
Sensor Fusion

• Given:
  – Position estimate $X \sim N(\mu=5; \sigma=1)$
  – Range estimate $R \sim N(\mu=3.2; \sigma=1.2)$

What is the best estimate AFTER incorporating $R$?

→ Kalman Filter

• Requires:
  – White Gaussian noise distribution for all measurements
  – Linear motion and measurement model
  – Position of the measured features
Feature-Based Localization

![Graph showing estimated, true, and range-based positions with parameters μ = 5; σ = 1 and μ = 5.8; σ = 1.2, and r = 3.2m.]}
Feature-Based Localization

$t=5$

- Estimated position: $\mu = 5.5; \sigma = 0.6$
- True position: $\mu = 5; \sigma = 1$
- Range-based position: $\mu = 5.8; \sigma = 1.2$
- Updated position: $r = 3.2m$
Feature-Based Localization

$\mu = 5.5; \sigma = 0.6$

$t = 5$
Feature-Based Localization

\[ t=6 \]

\[ \mu = 6.5; \sigma = 0.8 \]
The Kalman Filter Algorithm in 1D
Motivation for Stochastic Models and Estimation Methods

• A mathematical system model is necessarily an approximation of the real system: system structure is often appropriate but parameters are affected by uncertainties

• Non-deterministic sources of noise: we can neither control stochastic disturbances, nor get rid of stochastic sensor noise by calibration, nor model stochastic sources deterministically

• Sensors are noisy and can only partially observe the system state in all its details
Motivation for Stochastic Models and Estimation Methods

• The applicability of estimation methods goes way beyond localization problems

• To this purpose most diverse problems in environmental and civil engineering leverage Kalman filters for handling noisy processes and sensing

• Here are some examples I found in recent literature:
  – Improving accuracy of subsurface contaminant transport model
  – Improving in-situ monitoring of groundwater contamination
  – Identification of civil structural parameters
  – Improving predictive modeling of landslide deformation
  – Improving estimation of environmental performance variables in an acid gas removal process
Two Key Sources of Information

Stochastic models, estimation, and control

VOLUME 1

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Introduction to Autonomous Mobile Robots
Kalman Filter - Overview

- An **optimal recursive data processing algorithm**
- Combines all available measurement data, prior knowledge about system and measuring devices for producing a system state estimate with statistically minimized error

[From Siegwart and Nourbakhsh, 2004, adapted from Maybeck 1979]
Kalman Filter - Assumptions

- Linear system and measurement model
- White Gaussian system and measurement noise

Notes on noise:
- White: uncorrelated in time
- Gaussian: probability density of amplitude follows bell-shaped curve

\[
f(x) = \frac{1}{\sigma \sqrt{2\pi}} \exp\left(-\frac{(x - \mu)^2}{2\sigma^2}\right)
\]
A Simple 1D Positioning Example

\begin{align*}
t &= \text{time} \\
x &= \text{1D position} \\
\hat{x} &= \text{1D position estimate} \\
z &= \text{measurement or observation}
\end{align*}

\[ \hat{x}(t_1) = z_1 \]
\[ \sigma_x^2(t_1) = \sigma_z^2 \]

[From Maybeck, 1979]
Estimation Based on Static Measurements

- Second measurement taken at $t_2 \approx t_1$
- The smaller $\sigma$, the higher the certitude about the measurement

\[ \hat{x}(t_2) = z_2 \]
\[ \sigma^2_x(t_2) = \sigma^2_{z_2} \]

[From Maybeck, 1979]
Improving the Estimate Through Fusion

- Intuition: the smaller $\sigma$, and thus $\sigma^2$, the higher should be the weight in the fused estimate
- Estimate can be obtained as a weighted average $\mu$ of the individual measurement contributions

\[
\mu = \frac{1}{\frac{1}{\sigma_{Z1}^2} + \frac{1}{\sigma_{Z2}^2}} \left( \frac{1}{\sigma_{Z1}^2} \frac{Z1}{\sigma_{Z1}} + \frac{1}{\sigma_{Z2}^2} \frac{Z2}{\sigma_{Z2}} \right)
\]

\[
\frac{1}{\sigma^2} = \frac{1}{\sigma_{Z1}^2} + \frac{1}{\sigma_{Z2}^2}
\]

\[
\hat{x}(t_2) = \mu
\]

\[
\sigma_x(t_2) = \sigma
\]

$\sigma < \sigma_{Z1}$ and $\sigma < \sigma_{Z2}$

[From Maybeck, 1979]
Mean of the new Estimate

\[ \hat{x}(t_2) = \frac{1}{\sigma_{z1}^2 + \sigma_{z2}^2} z_1 + \frac{1}{\sigma_{z1}^2 + \sigma_{z2}^2} z_2 = \frac{\sigma_{z2}^2}{\sigma_{z1}^2 + \sigma_{z2}^2} z_1 + \frac{\sigma_{z1}^2}{\sigma_{z1}^2 + \sigma_{z2}^2} z_2 \]

\[ = \frac{\sigma_{z2}^2}{\sigma_{z1}^2 + \sigma_{z2}^2} z_1 + \frac{\sigma_{z1}^2}{\sigma_{z1}^2 + \sigma_{z2}^2} z_1 - \frac{\sigma_{z1}^2}{\sigma_{z1}^2 + \sigma_{z2}^2} z_1 + \frac{\sigma_{z1}^2}{\sigma_{z1}^2 + \sigma_{z2}^2} z_2 \]

\[ = \frac{\sigma_{z2}^2 + \sigma_{z1}^2}{\sigma_{z1}^2 + \sigma_{z2}^2} z_1 + \frac{\sigma_{z1}^2}{\sigma_{z1}^2 + \sigma_{z2}^2} (z_2 - z_1) \]

Kalman filter formulation

\[ \hat{x}(t_2) = \hat{x}(t_1) + K(t_2)[z_2 - \hat{x}(t_1)] \]

with \[ K(t_2) = \frac{\sigma_{z1}^2}{\sigma_{z1}^2 + \sigma_{z2}^2} \]

\[ \hat{x}(t_1) = z_1 \]
Variance of the new Estimate

\[
\frac{1}{\sigma_x^2(t_2)} = \frac{1}{\sigma_{z_1}^2} + \frac{1}{\sigma_{z_2}^2}
\]

\[
\sigma_x^2(t_2) = \sigma_{z_1}^2 \frac{\sigma_{z_2}^2}{\sigma_{z_1}^2 + \sigma_{z_2}^2} = \sigma_{z_2}^2 \frac{\sigma_{z_1}^2 + \sigma_{z_2}^2 - \sigma_{z_1}^2}{\sigma_{z_1}^2 + \sigma_{z_2}^2} \sigma_{z_1}^2 = \left(1 - \frac{\sigma_{z_1}^2}{\sigma_{z_1}^2 + \sigma_{z_2}^2}\right) \sigma_{z_1}^2
\]

\[
= \sigma_{z_1}^2 - \frac{\sigma_{z_1}^2}{\sigma_{z_1}^2 + \sigma_{z_2}^2} \sigma_{z_1}^2
\]

Kalman filter formulation

\[
\sigma_x^2(t_2) = \sigma_x^2(t_1) - K(t_2) \sigma_x^2(t_1)
\]

with \( K(t_2) = \frac{\sigma_{z_1}^2}{\sigma_{z_1}^2 + \sigma_{z_2}^2} \)

\[
\sigma_x^2(t_1) = \sigma_{z_1}^2
\]
Kalman Filter for Sensor Fusion

\[ \hat{x}(t_2) = \hat{x}(t_1) + K(t_2)[z_2 - \hat{x}(t_1)] \]

Mean

Prediction
Correction or Update

\[ \sigma^2_x(t_2) = \sigma^2_x(t_1) - K(t_2) \sigma^2_x(t_1) \]

Variance

\[ K(t_2) = \frac{\sigma^2_{z_1}}{\sigma^2_{z_1} + \sigma^2_{z_2}} \]

Kalman Gain

Function of the sensing precision
Estimation Considering Motion Dynamics

Consider a simple but noisy motion model: \[ \dot{x} = u + w \]

\( u = \) constant speed (controllable input)
\( w = \) Gaussian motion noise

\[ w = \sigma_w^2 \]

\( \sigma_k^2 : \) variance at timestep \( k \) (known)

\( \sigma_{k+1}^2 : \) variance at timestep \( k+1 \)

[From Siegwart and Nourbakhsh, 2004, adapted from Maybeck 1979]
Estimation Based on Motion Model

\[ \hat{x}_{k'} = \hat{x}_k + u[t_{k+1} - t_k] \]

- New mean position at timestep \( t_{k+1} \)
- Can be estimated with deterministic displacement from motion model

\[ \sigma_{k'}^2 = \sigma_k^2 + \sigma_w^2[t_{k+1} - t_k] \]

- New variance at timestep \( t_{k+1} \)
- Variance of noisy motion (constant over time) gets added (cumulated) to previous one
Fusing Motion Model Prediction with New Measurement - Mean

\[ \hat{x}_{k+1} = \hat{x}_{k'} + K_{k+1}(z_{k+1} - \hat{x}_{k'}) \]

Prediction based on motion model
Correction or update based on observation

with \( K_{k+1} = \frac{\sigma_{k'}^2}{\sigma_{k'}^2 + \sigma_z^2} \)

\( z_{k+1} \) : measurement at timestep \( k+1 \)
\( K_{k+1} \) : Kalman gain at timestep \( k+1 \)
\( \hat{x}_{k+1} \) : new estimate at timestep \( k+1 \) incorporating observation and prediction of motion model
\( \hat{x}_{k'} \) : estimate just before timestep \( k+1 \) based on prediction motion model
Fusing Motion Model Prediction with New Measurement - Variance

\[
\sigma_{k+1}^2 = \sigma_{k'}^2 - K_{k+1} \sigma_{k'}^2
\]

with \( K_{k+1} = \frac{\sigma_{k'}^2}{\sigma_{k'}^2 + \sigma_z^2} \)

\( K_{k+1} \): Kalman gain at timestep \( k+1 \)
\( \sigma_{k+1}^2 \): variance at timestep \( k+1 \) incorporating correction from observation and prediction of motion model
\( \sigma_{k'}^2 \): variance just before timestep \( k+1 \) based on prediction motion model
\( \sigma_z^2 \): variance of the sensor measurement (constant over time)
Kalman Filter for Sensor and Motion Model Fusion

\[ \hat{x}_{k+1} = \hat{x}_{k'} + K_{k+1} (z_{k+1} - \hat{x}_{k'}) \]

Prediction \hspace{1cm} Correction or Update

\[ \sigma_{k+1}^2 = \sigma_{k'}^2 - K_{k+1} \sigma_{k'}^2 \]

Mean

Variance

Kalman Gain

Function of the motion model and sensing precision

\[ K_{k+1} = \frac{\sigma_{k'}^2}{\sigma_{k'}^2 + \sigma_z^2} \]
Kalman Filter - Some Extreme Cases

\[
\hat{x}_{k+1} = \hat{x}_{k'} + K_{k+1}(z_{k+1} - \hat{x}_{k'})
\]

\[
\sigma^2_{k+1} = \sigma^2_{k'} - K_{k+1} \sigma^2_{k'}
\]

\[
K_{k+1} = \frac{\sigma^2_{k'}}{\sigma^2_{k'} + \sigma^2_z}
\]

\[
\sigma^2_z \rightarrow \infty: \text{new measurement extremely noisy, does not add information, then } K_{k+1} \rightarrow 0 \text{ new estimate based exclusively on motion model (both mean and variance)}
\]

\[
\sigma^2_w \rightarrow \infty, \text{ then } \sigma^2_{k'} \rightarrow \infty \text{ (see. s. 23): motion model does not add information, then } K_{k+1} \rightarrow 1 \text{ new estimate based exclusively on new observation}
\]

\[
\sigma^2_{k'} \rightarrow 0, \text{ motion model is deterministic and perfectly reproducing the reality, then } K_{k+1} \rightarrow 0, \text{ new measurement can be disregarded since model is giving a perfect estimate}
\]
Non-Deterministic Error Sources in Wheel-Based Odometry
Nondeterministic Error Sources

- Variation of the contact point of the wheel
- Unequal floor contact (e.g., wheel slip, nonplanar surface)

- Wheels cannot be assumed to roll perfectly
- Measured encoder values do not perfectly reflect the actual motion
- Pose error is cumulative and incrementally increases
- Probabilistic modeling for assessing quantitatively the error
Odometric Error Types

• Range error: sum of the wheel movements

• Turn error: difference of wheel motion

• Drift error: difference between wheel errors lead to heading error
Pose Variation During $\Delta t$

\[
\Delta s = \frac{\Delta s_r + \Delta s_l}{2}
\]

\[
\begin{align*}
\Delta x &= \Delta s \cos(\theta + \frac{\Delta \theta}{2}) \\
\Delta y &= \Delta s \sin(\theta + \frac{\Delta \theta}{2}) \\
\Delta \theta &= \frac{2}{b} \frac{\Delta s_r - \Delta s_l}{b}
\end{align*}
\]

\[
p = \begin{bmatrix} x \\ y \\ \theta \end{bmatrix} \quad \overset{t'=t+\Delta t}{\rightarrow} \quad p' = \begin{bmatrix} x' \\ y' \\ \theta' \end{bmatrix}
\]

\[
p' = \begin{bmatrix} x \\ y \\ \theta \end{bmatrix} + \begin{bmatrix} \Delta s \cos(\theta + \Delta \theta / 2) \\ \Delta s \sin(\theta + \Delta \theta / 2) \\ \Delta \theta \end{bmatrix} = \begin{bmatrix} x \\ y \\ \theta \end{bmatrix} + \begin{bmatrix} \frac{\Delta s_r + \Delta s_l}{2} \cos(\theta + \frac{\Delta s_r - \Delta s_l}{2b}) \\ \frac{\Delta s_r + \Delta s_l}{2} \sin(\theta + \frac{\Delta s_r - \Delta s_l}{2b}) \\ \frac{\Delta s_r - \Delta s_l}{b} \end{bmatrix}
\]

\[
b = \text{inter-wheel distance} \\
\Delta s_r = \text{traveled distance right wheel} \\
\Delta s_l = \text{traveled distance left wheel} \\
\Delta \theta = \text{orientation change of the vehicle}
\]
Noise modeling

Model error in each dimension with a Gaussian $x \rightarrow \bar{x}, \sigma_x; y \rightarrow \bar{y}, \sigma_y; \theta \rightarrow \bar{\theta}, \sigma_\theta$

$$\Sigma_p = \begin{bmatrix}
\sigma_{xx}^2 & \sigma_{xy}^2 & \sigma_{x\theta}^2 \\
\sigma_{yx}^2 & \sigma_{yy}^2 & \sigma_{y\theta}^2 \\
\sigma_{\theta x}^2 & \sigma_{\theta y}^2 & \sigma_{\theta\theta}^2
\end{bmatrix}$$

Assumptions:

- Covariance matrix $\Sigma_p$ at the beginning is known
- Errors of the two individual wheels are independent
- Errors are independent of direction of motion
- Errors are proportional to the distance traveled ($k_r, k_l$ model parameters)

$$\sum_{\Delta} = \text{cov}(\Delta s_r, \Delta s_l) = \begin{bmatrix}
k_r |\Delta s_r| & 0 \\
0 & k_l |\Delta s_l|
\end{bmatrix} = \begin{bmatrix}
\sigma_{s_r}^2 & 0 \\
0 & \sigma_{s_l}^2
\end{bmatrix}$$
Actuator Noise $\rightarrow$ Pose Noise

- How is the actuator noise (2D) propagated to the pose (3D)?

\[ \Sigma_\Delta = \begin{bmatrix} \sigma_{s_r}^2 & 0 \\ 0 & \sigma_{s_l}^2 \end{bmatrix} \]

\[ \begin{bmatrix} \sigma_{s_r}^2 \\ \sigma_{s_l}^2 \end{bmatrix} \rightarrow \begin{bmatrix} \sigma_x^2 \\ \sigma_y^2 \\ \sigma_\theta^2 \end{bmatrix} \]

\[ \begin{bmatrix} \sigma_{xx}^2 & \sigma_{xy}^2 & \sigma_{x\theta}^2 \\ \sigma_{yx}^2 & \sigma_{yy}^2 & \sigma_{y\theta}^2 \\ \sigma_{\theta x}^2 & \sigma_{\theta y}^2 & \sigma_{\theta\theta}^2 \end{bmatrix} = \Sigma_p \]

- 1D to 1D example \( N(\mu_{s_r}, \sigma_{s_r}) \rightarrow N(\mu_x, \sigma_x) \)

- We need to linearize $\rightarrow$ Taylor Series

\[ x \approx f(\Delta s_r) \bigg|_{\Delta s_r = \mu_{s_r}} \approx f(\Delta s_r) + \frac{1}{1!} \frac{\partial f}{\partial \Delta s_r} (\Delta s_r - \mu_{s_r}) + \frac{1}{2!} \frac{\partial^2 f}{\partial \Delta s_r^2} (\Delta s_r - \mu_{s_r})^2 + \cdots \]
Actuator Noise $\rightarrow$ Pose Noise

$$
\begin{bmatrix}
  x \\
  y \\
  \theta
\end{bmatrix} =
\begin{bmatrix}
  f_1(\Delta s_r, \Delta s_l) \\
  f_2(\Delta s_r, \Delta s_l) \\
  f_3(\Delta s_r, \Delta s_l)
\end{bmatrix} =
\begin{bmatrix}
  \frac{\Delta s_r + \Delta s_l}{2} \cos(\theta + \frac{\Delta s_r - \Delta s_l}{2b}) \\
  \frac{\Delta s_r + \Delta s_l}{2} \sin(\theta + \frac{\Delta s_r - \Delta s_l}{2b}) \\
  \frac{\Delta s_r - \Delta s_l}{b}
\end{bmatrix}
$$

$$
\frac{\partial f_1}{\partial \Delta s_r} = \frac{\partial}{\partial \Delta s_r} \left( \frac{\Delta s_r + \Delta s_l}{2} \right) \cos(\theta + \frac{\Delta s_r - \Delta s_l}{2b}) + \left( \frac{\Delta s_r + \Delta s_l}{2} \right) \frac{\partial}{\partial \Delta s_r} \left[ \cos(\theta + \frac{\Delta s_r - \Delta s_l}{2b}) \right]
$$

$$
\frac{\partial f_1}{\partial \Delta s_r} = \frac{1}{2} \cos(\theta + \frac{\Delta s_r - \Delta s_l}{2b}) + \left( \frac{\Delta s_r + \Delta s_l}{2} \right) \left[ - \sin(\theta + \frac{\Delta s_r - \Delta s_l}{2b}) \right] \frac{\partial}{\partial \Delta s_r} \left( \theta + \frac{\Delta s_r - \Delta s_l}{2b} \right)
$$

$$
\frac{\partial f_1}{\partial \Delta s_r} = \frac{1}{2} \cos(\theta + \frac{\Delta s_r - \Delta s_l}{2b}) - \left( \frac{\Delta s_r + \Delta s_l}{2} \right) \left[ \sin(\theta + \frac{\Delta s_r - \Delta s_l}{2b}) \frac{1}{2b} \right]
$$

$$
\frac{\partial f_1}{\partial \Delta s_r} = \frac{1}{2} \cos(\theta + \frac{\Delta s_r - \Delta s_l}{2b}) - \left( \frac{\Delta s_r + \Delta s_l}{4b} \right) \sin(\theta + \frac{\Delta s_r - \Delta s_l}{2b})
$$
Actuator Noise $\rightarrow$ Pose Noise

$$x \approx f_1(\Delta s_r) \bigg|_{\Delta s_r = \mu_{s_r}} \approx f_1(\Delta s_r) + \frac{\partial f_1}{\partial \Delta s_r} (\Delta s_r - \mu_{s_r})$$
Actuator Noise → Pose Noise

\[ x \approx f_1(\Delta s_r) \bigg|_{\Delta s_r = \mu_{s_r}} \approx f_1(\Delta s_r) + \frac{\partial f_1}{\partial \Delta s_r} (\Delta s_r - \mu_{s_r}) \]

\[ x \approx f_1(\Delta s_l) \bigg|_{\Delta s_l = \mu_{s_l}} \approx f_1(\Delta s_l) + \frac{\partial f_1}{\partial \Delta s_l} (\Delta s_l - \mu_{s_l}) \]
Actuator Noise $\rightarrow$ Pose Noise

\[ x \approx f_1(\Delta s_r) \bigg|_{\Delta s_r = \mu_{s_r}} \approx f_1(\Delta s_r) + \frac{\partial f_1}{\partial \Delta s_r} (\Delta s_r - \mu_{s_r}) \]

\[ x \approx f_1(\Delta s_l) \bigg|_{\Delta s_l = \mu_{s_l}} \approx f_1(\Delta s_l) + \frac{\partial f_1}{\partial \Delta s_l} (\Delta s_l - \mu_{s_l}) \]

\[ y \approx f_2(\Delta s_r) \bigg|_{\Delta s_r = \mu_{s_r}} \approx f_2(\Delta s_r) + \frac{\partial f_2}{\partial \Delta s_r} (\Delta s_r - \mu_{s_r}) \]

\[ y \approx f_2(\Delta s_l) \bigg|_{\Delta s_l = \mu_{s_l}} \approx f_2(\Delta s_l) + \frac{\partial f_2}{\partial \Delta s_l} (\Delta s_l - \mu_{s_l}) \]
Actuator Noise → Pose Noise

\[ x \approx f_1(\Delta s_r) \bigg|_{\Delta s_r=\mu_{s_r}} \approx f_1(\Delta s_r) + \frac{\partial f_1}{\partial \Delta s_r}(\Delta s_r - \mu_{s_r}) \]

\[ x \approx f_1(\Delta s_l) \bigg|_{\Delta s_l=\mu_{s_l}} \approx f_1(\Delta s_l) + \frac{\partial f_1}{\partial \Delta s_l}(\Delta s_l - \mu_{s_l}) \]

\[ y \approx f_2(\Delta s_r) \bigg|_{\Delta s_r=\mu_{s_r}} \approx f_2(\Delta s_r) + \frac{\partial f_2}{\partial \Delta s_r}(\Delta s_r - \mu_{s_r}) \]

\[ \ldots \]

\[ F_{\Delta rl} = \begin{bmatrix} \frac{\partial f_1}{\partial \Delta s_r} & \frac{\partial f_1}{\partial \Delta s_l} \\ \frac{\partial f_2}{\partial \Delta s_r} & \frac{\partial f_2}{\partial \Delta s_l} \\ \frac{\partial f_3}{\partial \Delta s_r} & \frac{\partial f_3}{\partial \Delta s_l} \end{bmatrix} \text{ Jacobian} \]

\[ \sum_{\Delta} = \begin{bmatrix} k_r |\Delta s_r| & 0 \\ 0 & k_l |\Delta s_l| \end{bmatrix} \text{ Covariance matrix for wheel noise} \]

\[ \Sigma_{\Delta rl} = F_{\Delta rl} \Sigma_{\Delta} F_{\Delta rl}^T \text{ Covariance matrix for the noise propagated to the pose} \]
Actuator Noise $\rightarrow$ Pose Noise

How does the pose covariance $\Sigma_p$ evolve over time?

- Initial covariance of vehicle at $t=0$: \[
\Sigma^{(t=0)}_p = \begin{bmatrix}
\sigma^2_{xx} & \sigma^2_{xy} & \sigma^2_{x\theta} \\
\sigma^2_{yx} & \sigma^2_{yy} & \sigma^2_{y\theta} \\
\sigma^2_{\theta x} & \sigma^2_{\theta y} & \sigma^2_{\theta\theta}
\end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\]

- Additional noise at each time step $\Delta t$: $\Sigma_{\Delta rl} = F_{\Delta rl} \Sigma \Delta F_{\Delta rl}^T$

- Covariance at $t=1\Delta t$: $\Sigma^{(t=1\Delta t)}_p = \Sigma^{(t=0)}_p + \Sigma_{\Delta rl} = \Sigma_{\Delta rl}$

- Covariance at $t=2\Delta t$: \[
\Sigma^{(t=2\Delta t)}_p = F_p \Sigma^{(t=1\Delta t)}_p F_p^T + F_{\Delta rl} \Sigma \Delta F_{\Delta rl}^T
\]

$F_p = \begin{bmatrix}
\frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} & \frac{\partial f_1}{\partial \theta} \\
\frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} & \frac{\partial f_2}{\partial \theta} \\
\frac{\partial f_3}{\partial x} & \frac{\partial f_3}{\partial y} & \frac{\partial f_3}{\partial \theta}
\end{bmatrix}$
Actuator Noise → Pose Noise

Algorithm

Precompute:

- Determine actuator noise $\Sigma_{\Delta}$
- Compute mapping actuator-to-pose noise incremental $F_{\Delta r l}$
- Compute mapping pose propagation noise over step $F_p$

Initialize:

- Initialize $\Sigma_p^{(t=0)} = [0]$

Iterate:

$$\Sigma_p^{(t=(k+1)\Delta t)} = F_p \Sigma_p^{(t=k\Delta t)} F_p^T + F_{\Delta r l} \Sigma_{\Delta} F_{\Delta r l}$$
Classical 2D Representation

Ellipses: typical $3\sigma$ bounds

Courtesy of R. Siegwart and R. Nourbakhsh
Conclusion
Take Home Messages

- Odometry is an absolute positioning method using only proprioceptive sensors but affected by a cumulative error.
- Localization error in odometry can be both deterministic and non-deterministic.
- Deterministic errors can be mitigated by calibration, non-deterministic errors can be modeled and taken into account, typically leveraging the kinematic forward model of the vehicle.
- Feature-based localization is a way to compensate odometry limitations by leveraging exteroceptive sensors in addition to proprioceptive ones.
- Estimation methods, including Kalman filter techniques are applicable to a large variety of problems involving noisy processes and noisy sensing, not just localization.
- A Kalman filter is a computationally efficient, optimal recursive data processing algorithm that allows fusion of multiple estimates coming from either process or sensing models.
- A Kalman filter assumes linear motion and sensor models characterized by white Gaussian noise: this is not always fulfilled.
Additional Literature – Week 10

Pointers

http://www.probabilistic-robotics.org/

Books