Signals, Instruments, and Systems – W3

Introduction to Signal Processing – Sampling, Reconstruction, and Aliasing
Motivation from Week 1 Lecture

Hilgghted blocks are those mainly leveraging the content of this lecture.
Sampling
Analog-Digital Converter (ADC)

• Transforms continuous analog signal into series of values
• Two key elements
  – **Sampling** (in time)
  – **Quantization** (of values)

\[ y[n] = 0 \quad 0 \quad -2 \quad -4 \quad -2 \quad 0 \quad 4 \quad 8 \quad 10 \quad 10 \]
Periodic Train of Dirac Impulses

\[ x(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT) \]
**Sampling in Time Domain**

\[ x(t) \]

**Periodic train of Dirac impulses**

\[ p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT) \]

\[ x_p(t) = x(t) p(t) = \sum_{n=-\infty}^{\infty} x(t) \delta(t - nT) = \sum_{n=-\infty}^{\infty} x(nT) \delta(t - nT) \]

\[ T = \text{sampling period} \]

\[ f_s = 1/T = \text{sampling frequency} \]
Sampling in the Frequency Domain

• It would be nice to understand what does it mean sampling in the frequency domain so that we can leverage this representation for further reasoning (e.g., filter design)

• Multiplication in time domain means convolution in frequency domain but …

• How does look like a periodic train of Dirac impulses in the frequency domain?
Complex Coefficients for Arbitrary Period

Fourier series

\[ f(t) = \sum_{n=-\infty}^{\infty} C_n e^{in\omega_0 t} \]

Fourier coefficients for T periodic function

\[ C_n = \frac{1}{T} \int_{-T/2}^{T/2} f(t) e^{-in\omega_0 t} \, dt \]

Rem: \( C_n = |C_n| e^{i\varphi} \)

Magnitude: |\( C_n | \)

Phase: \( \varphi \)

From W2 (s.18)
Fourier Transform

Non-unitary, angular frequency notation

\[
F(\omega) = \int_{-\infty}^{+\infty} f(t) e^{-i\omega t} \, dt
\]

\[
f(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} F(\omega) e^{i\omega t} \, d\omega
\]

Notes:
- \(\omega = 2\pi \xi \rightarrow \) obtained from unitary, ordinary frequency transform with \(\xi = \omega/2\pi\)
- F can be replaced with \(f^\wedge\)
- In electrical engineering \(i\) is substituted by \(j\) (“i” booked for current)
- Often, in order to emphasize the frequency response aspect, the imaginary aspect of the transform is emphasized: \(F(\omega), F(i\omega), \text{or } F(j\omega)\) are all equivalent notations

From W2 (s.23)
FT of a Dirac Delta Function (or Dirac Impulse)

\[ f(t) = \delta(t) \]

\[ \hat{f}(\xi) = 1 \]
FT for Periodic Signals

• Although it generalizes to aperiodic signals, the FT can be also applied to periodic signals.

• We can derive the FT of a periodic signal from its Fourier series (which have been developed for periodic signals, see W2).

• The FT of a periodic Dirac train of impulses consists of a periodic train of impulses in the frequency domain as well, with the area of the impulses proportional to the Fourier series coefficients.

From [Oppenheim et al., 1997]
FT for Periodic Signals

(assume: period T, \( \omega_0 = 2\pi / T \))

\[
X(\omega) = 2\pi \delta(\omega - \omega_0)
\]

\[
x(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} 2\pi \delta(\omega - \omega_0) \, e^{i\omega t} \, d\omega
\]

\[
x(t) = e^{i\omega_0 t}
\]

Linear combination of impulses equally spaced in frequency:

\[
X(\omega) = \sum_{n=-\infty}^{+\infty} 2\pi C_n \delta(\omega - n\omega_0)
\]

\[
x(t) = \sum_{n=-\infty}^{+\infty} C_n e^{i n \omega_0 t}
\]
FT for Periodic Impulse Train
(assume: period T, $\omega_0 = 2\pi/T$)

Now consider the periodic impulse train of before:

$$x(t) = \sum_{n=-\infty}^{+\infty} \delta(t - nT)$$

Calculate the coefficient of its Fourier series:

$$C_n = \frac{1}{T} \int_{-T/2}^{+T/2} \delta(t) e^{-in\omega_0 t} \, dt = \frac{1}{T}$$

This means that each Fourier coefficient of the periodic impulse train has the same value; insert $C_n$ in previous expression (s. 12):

$$X(\omega) = \sum_{n=-\infty}^{+\infty} 2\pi C_n \delta(\omega - n\omega_0)$$

$$X(\omega) = \sum_{n=-\infty}^{+\infty} \frac{2\pi}{T} \delta(\omega - n\omega_0) = \frac{2\pi}{T} \sum_{n=-\infty}^{+\infty} \delta(\omega - n\omega_0)$$
Train of Dirac Impulses

\[ x(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT) \]

\[ X(\omega) = \sum_{k=-\infty}^{\infty} \frac{2\pi}{T} \delta(\omega - k\omega_s) \]

\[ \omega_s = \frac{2\pi}{T} \]

Sampling angular frequency

Time domain

Frequency domain
Sampling in Frequency Domain

Time domain

\[ x_p(t) = x(t)p(t) \]

Frequency domain

\[ X_p(\omega) = \frac{1}{2\pi} X(\omega) * P(\omega) \]

\[ P(\omega) = \frac{2\pi}{T} \sum_{k=-\infty}^{\infty} \delta(\omega - k\omega_s) \]

\[ X_p(\omega) = \frac{1}{T} \sum_{k=-\infty}^{\infty} X(\omega) * \delta(\omega - k\omega_s) \]

\[ = \frac{1}{T} \sum_{k=-\infty}^{\infty} X(\omega - k\omega_s) \]

FT properties

From s.14

Note: see also W2 ss. 45-47 as examples for this operation
Sampling a Band-Limited Signal

\[ X(\omega) \rightarrow \text{spectrum of signal } x(t) \]
with highest frequency < \( \omega_m \)

\[
P(\omega) = \frac{2\pi}{T} \sum_{k=-\infty}^{\infty} \delta(\omega - k\omega_s)
\]

Angular sampling frequency
\( \omega_s > 2 \omega_m \)

\[
X_p(\omega) = \frac{1}{2\pi} X(\omega) * P(\omega)
= \frac{1}{T} \sum_{k=-\infty}^{\infty} X(\omega - k\omega_s)
\]
Sampling a Band-Limited Signal

$X(\xi) \rightarrow$ spectrum of signal $x(t)$ with highest frequency $< 2 \text{ kHz}$

Sampling frequency: $5 \text{ kHz} > 2 \times 2 \text{ kHz}$
Sampling a Band-Limited Signal

$X(\xi) \rightarrow$ spectrum of signal $x(t)$ with highest frequency $< 3 \text{ kHz}$

$X(f)$

-3 kHz $\rightarrow$ 0 $\rightarrow$ 3 kHz

$f [\text{Hz}]$

$P(f)$

$2\pi/T$

Sampling frequency: $5 \text{ kHz} < 2 \times 3 \text{ kHz}$
Original Signal

\[ f(t) = \sin(2\pi t) + 0.4 \sin(2\pi \cdot 2t) + 0.2 \sin(2\pi \cdot 5t) \]
Too Few Samples (1Hz)

→ Data is lost
Too Many Samples (100 Hz)

→ Redundant data
→ Increase of data size
Minimal Possible Sampling (> 10 Hz)
Nyquist–Shannon Theorem

- If a function $x(t)$ contains no frequencies higher than $B$ Hz, it is completely determined by giving its coordinates at a series of points spaced $1/(2B)$ seconds apart.
- Sampling frequency must be at least two times greater than the maximal signal frequency.
Sampling in Practice

• Sampling frequency two times greater than maximal frequency is the limit

• Example (parsimony principle applied): audio CD, sampling at 44.1 kHz since maximal hearable frequency is 20 kHz

• If affordable, try to use a sampling frequency 10 times greater than the maximal frequency (help all sorts of filtering and reconstruction processes)
Signal Reconstruction
If there is no overlap between the shifted spectra the signal $x_r(t)$ can be perfectly reconstructed from $x(t)$.

Spectrum of original signal

Spectrum of sampled signal

Sampling angular frequency $\omega_s > 2 \omega_m$

Filtering

Filter cut-off angular frequency $\omega_m < \omega_c < (\omega_s - \omega_m)$

$$X_r(\omega) = X_p(\omega)H(\omega)$$

Spectrum of reconstructed signal
\[ p(t) = \sum_{n=-\infty}^{+\infty} \delta(t - nT) \]

\[ x(t) \rightarrow x_p(t) \rightarrow H(\omega) \rightarrow x_r(t) \]

If there is overlap between the shifted spectra the signal \( x_r(t) \) cannot be perfectly reconstructed from \( x(t) \)

spectrum of original signal

spectrum of sampled signal

sampling frequency
\[ f_s < 2 f_m \]

filtering

filter cut-off frequency \( f_c \)
\[ f_m < f_c < (f_s - f_m) \]

\[ X_r(\omega) = X_p(\omega)H(\omega) \]

spectrum of reconstructed signal
Time Domain Interpretation of Signal Reconstruction

\[ p(t) = \sum_{n=-\infty}^{+\infty} \delta(t - nT) \]

- Multiplication with a Low-Pass Filter (LPF) in the frequency domain
- The LPF interpolates the samples assuming \( x(t) \) contains no energy at frequencies > \( \omega_c \) (\( \omega_c \) = cutoff angular frequency)

\[ x_r(t) = x_p(t) * h(t) \]

with \( h(t) = \frac{T \sin(\omega_c t)}{\pi t} \)

Note: see also W2 ss. 45-47 as examples for this operation
Signal Reconstruction in Practice

1. Whittaker-Shannon interpolation (band-limited interpolation):

- Signal has to be band limited
  (i.e. Fourier transform for frequencies greater than B equal 0)
- The sampling rate must exceed twice the bandwidth, 2B, i.e. $f_s > 2B$

Assume: $\omega_c = \frac{\omega_s}{2} = \frac{2\pi}{2T} = \frac{\pi}{T}$ in s. 28, $\text{sinc}(x) = \frac{\sin(\pi x)}{\pi x}$ Normalized sinc function

$$x_r(t) = \sum_{n=-\infty}^{\infty} x(nT) \cdot \text{sinc}\left(\frac{t-nT}{T}\right)$$

$$x_r(t) = \left(\sum_{n=-\infty}^{\infty} x(nT) \cdot \delta(t - nT)\right) \ast \text{sinc}\left(\frac{t}{T}\right) \quad \text{(Alternative equivalent formulation)}$$
Graphic Illustration of Time-Domain Interpolation

Original CT signal

After sampling

After passing the LPF (Low Pass Filter)

From Prof. A. S. Willsky, Signals and Systems course
Signal Reconstruction in Practice

2. Zero-order hold

3. First-order hold (linear interpolation)

From Prof. A. S. Willsky, Signals and Systems course
Reconstruction Summary – Time Domain

The reconstructed signal $x_r(t)$ is obtained through a convolution between the sampled signal $x_p(t)$ with period $T$ and one of the following three interpolation functions $h(t)$.

1. Zero-order hold (ZOH)

2. First-order hold (FOH)

3. Whittaker-Shannon

From Prof. A. S. Willsky, Signals and Systems course
Reconstruction Summary – Frequency Domain

The spectrum of the reconstructed signal $X_r(\omega)$ is obtained through a multiplication between the spectrum of the sampled signal $X_p(\omega)$ with angular sampling frequency $\omega_s$ and one of the following three low-pass filters.

From Prof. A. Oppenheim, Signals and Systems course
Aliasing
No Problems in Reconstruction

\[ H(f) = \text{rect}(\frac{f}{f_s}) \]

\[ X_s(f) \]

\[ X(f) \]
Reconstruction Problems

Overlapping

Alias

$X(f)$

$X_A(f)$
Harmonics

- Time [s]
- 1Hz
- 2Hz
- 3Hz
- 4Hz
Harmonics

Fundamental Frequency

0 \quad 1

1/2

1/3

1/4

1/5

1/6

1/7
La-Tone (440 Hz) sampled at 44.1 kHz (CD standard)

Rem: s. 6, W2, max audible signals by humans 20 kHz
La-Tone (440 Hz) sampled at 2 kHz

Reduced sampling frequency: 2 kHz
La-Tone (440 Hz) sampled at 2 kHz without filtering

Original signal with aliasing effect (A)

Repeated & shifted spectrum through sampling
La-Tone (440 Hz) sampled at 2 kHz

- Reduced sampling frequency: 2 kHz
- Anti-alias filter: desired cut-off frequency 1 kHz
- Actual high-order digital filter (see next week)
La-Tone (440 Hz) sampled at 2 kHz filtered at 1 kHz

Original signal without aliases through anti-alias filtering

Repeated & shifted spectrum through sampling will be eliminated through low-pass filtering at signal reconstruction stage; however, harmonics above 1 kHz also cut by the anti-alias filter.
Aliasing Audio Examples

Original sound  Aliases 4 kHz  Correct sampling 4 kHz

Note: for instance cymbals (8-16 kHz) fully cut off, no matter what
Moiré Pattern
Aliasing Video Examples

https://www.youtube.com/watch?v=jHS9JGkEOmA
https://www.youtube.com/watch?v=R-IVw8OKjvQ
Conclusion
Take Home Messages

• Sampling: multiplication of the signal with a periodic train of Dirac impulses results in a repeated shifted spectrum in the frequency domain

• Multiple signal reconstructions algorithms are available, more or less computationally expensive; they can all be represented by low-pass filters in the frequency domain

• Aliasing: higher frequency “folded back” on the original spectrum -> prevent proper signal reconstruction

• When you sample a signal …
  – Make sure you know what the maximum frequency $f_{\text{max}}$ is or enforce it through an anti-alias low-pass filter
  – Make sure you sample at $f_s > 2 f_{\text{max}}$ (Nyquist-Shannon theorem)
Additional Literature – Week 3

• Books: