

Signals, Instruments, and Systems – W2
An Introduction to Signal
Processing – Fourier Series,
Fourier Transforms, and
Convolution

Outline

- Some typical signals and values
- Fourier series
- Continuous and discrete Fourier transforms
- Continuous and discrete convolution

Typical Waves and Signals

Waves

“A wave is a disturbance that propagates through space and time, usually with transference of energy.”

Wave function:

$$f(x, t) = \gamma \sin(kx - \omega t)$$

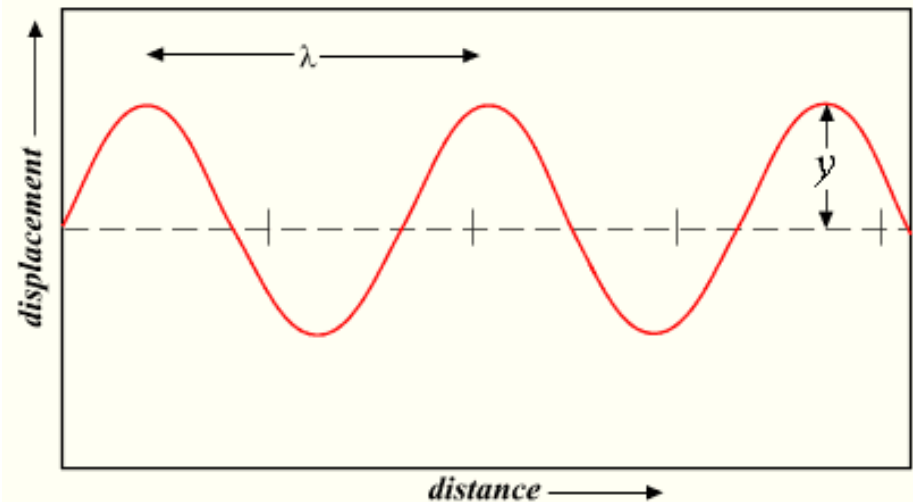
$$k = \frac{2\pi}{\lambda} \quad \text{“wave number”}$$

$$\omega = 2\pi f \quad \text{“angular frequency”}$$

$$\lambda = \frac{v}{f} \quad \text{“wave length”}$$

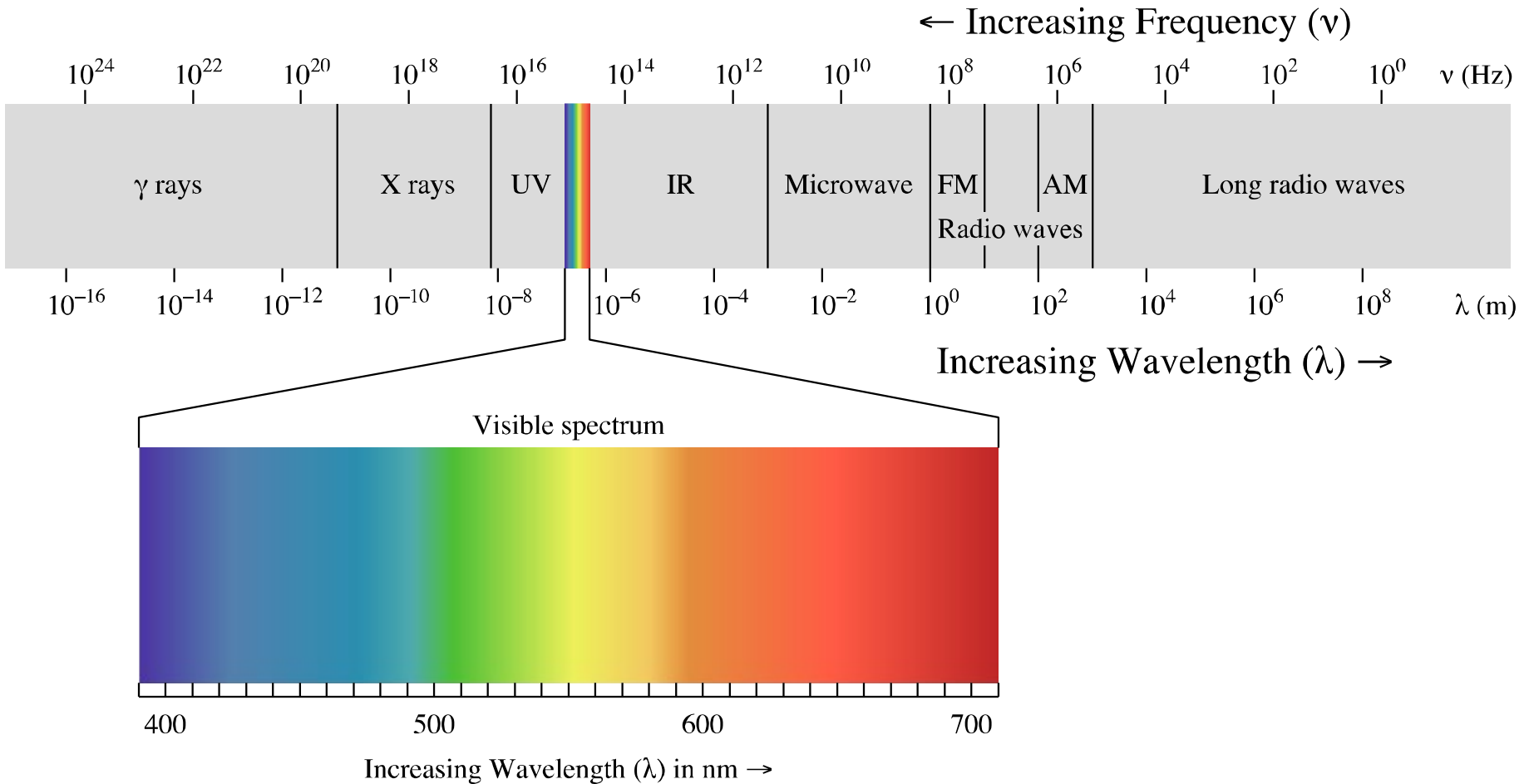


Wave



$\lambda = \text{wavelength}$ $v = \text{speed in x direction}$
 $\gamma = \text{amplitude}$

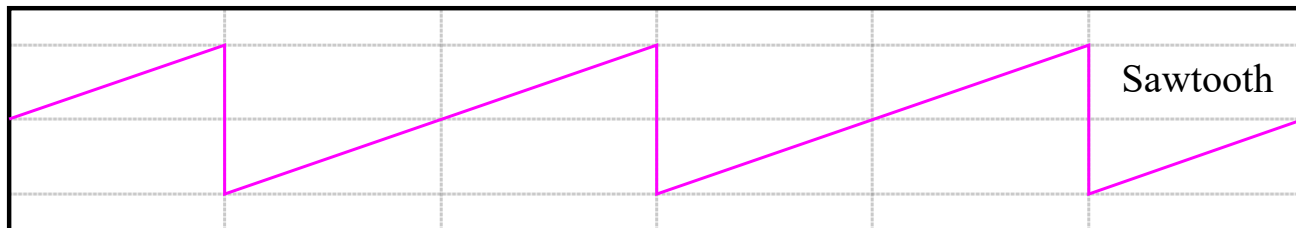
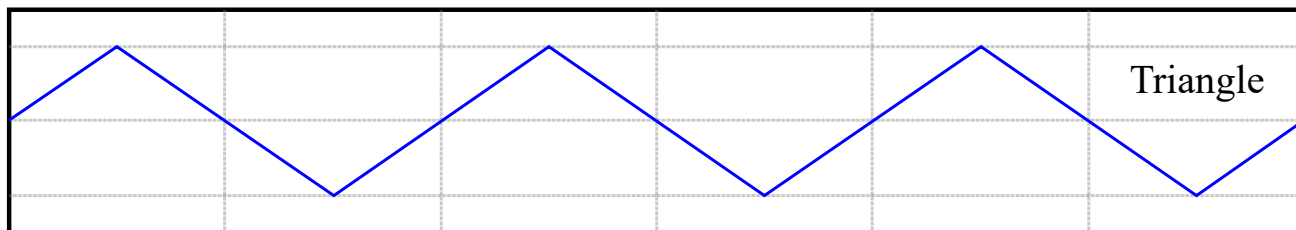
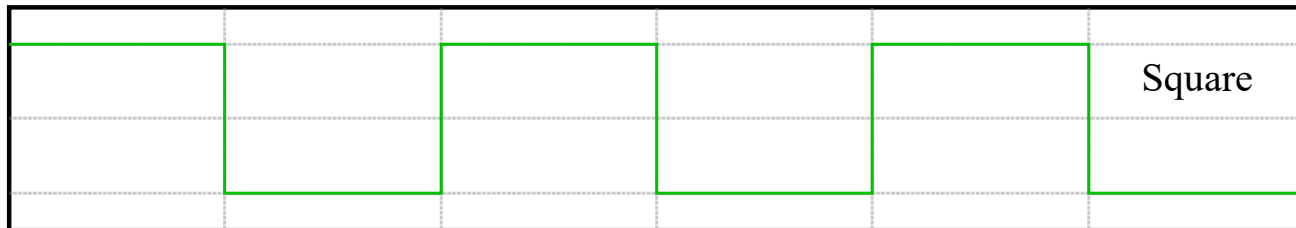
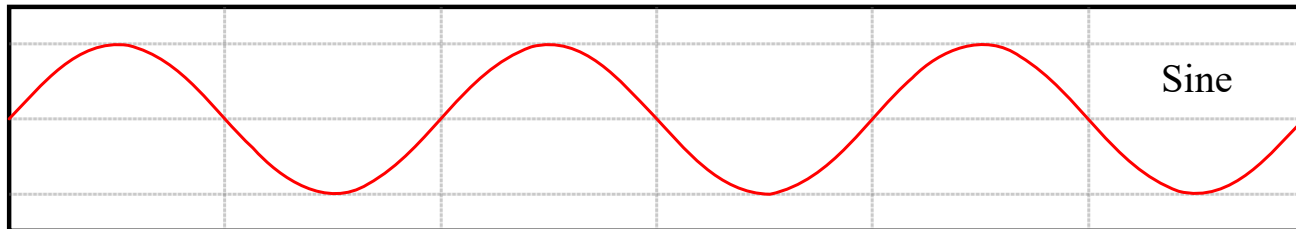
Electromagnetic Spectrum



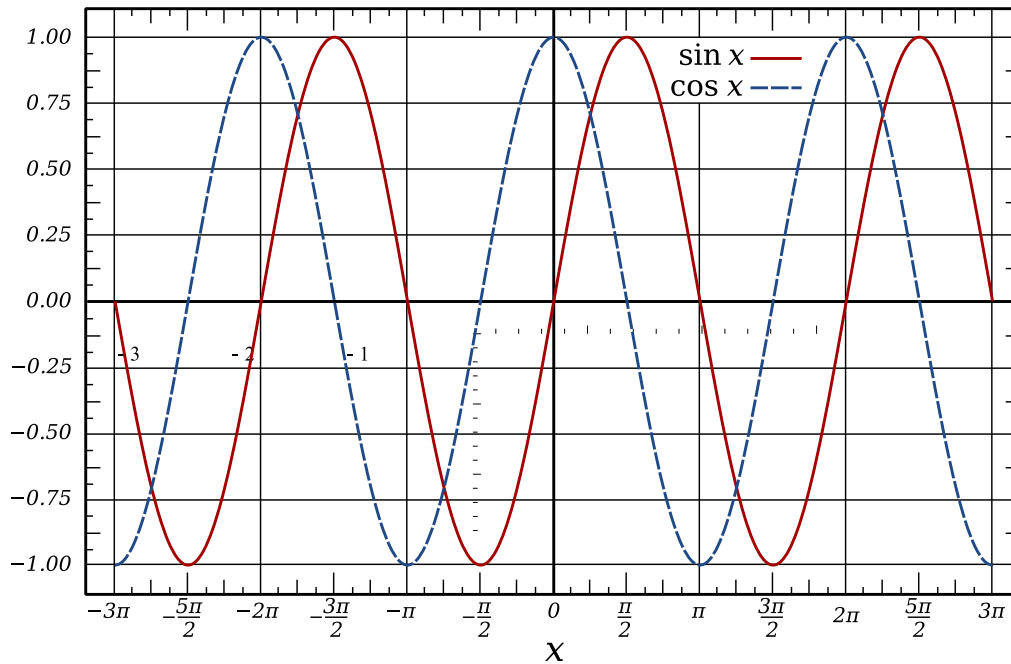
Common Frequencies

- Car motor: ~ 50 Hz (3000 rpm)
- Power lines: 50 Hz
- Ear: 20 Hz – 20 kHz
- Ultrasound: 20 kHz – 200 MHz
- Medical sonograph: ~ 2 -18 MHz
- Watch quartz: 32 kHz
- CPU: 2-3 GHz
- GSM: 0.9/1.8 GHz

Standard Waveforms



Sinusoid (Sine Wave) in Time



A : amplitude

ω : angular frequency

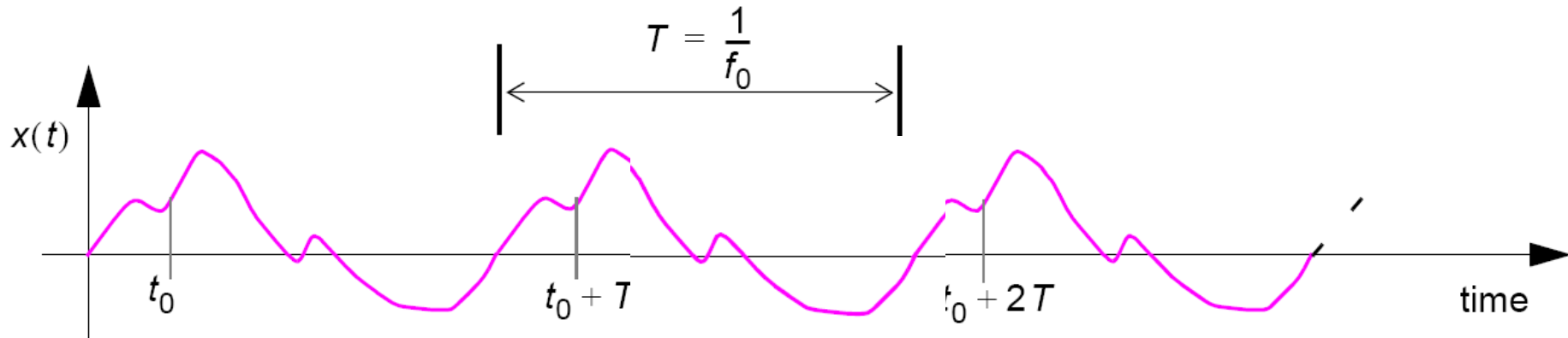
θ : phase

$$y(t) = A \cdot \sin(\omega t + \theta)$$

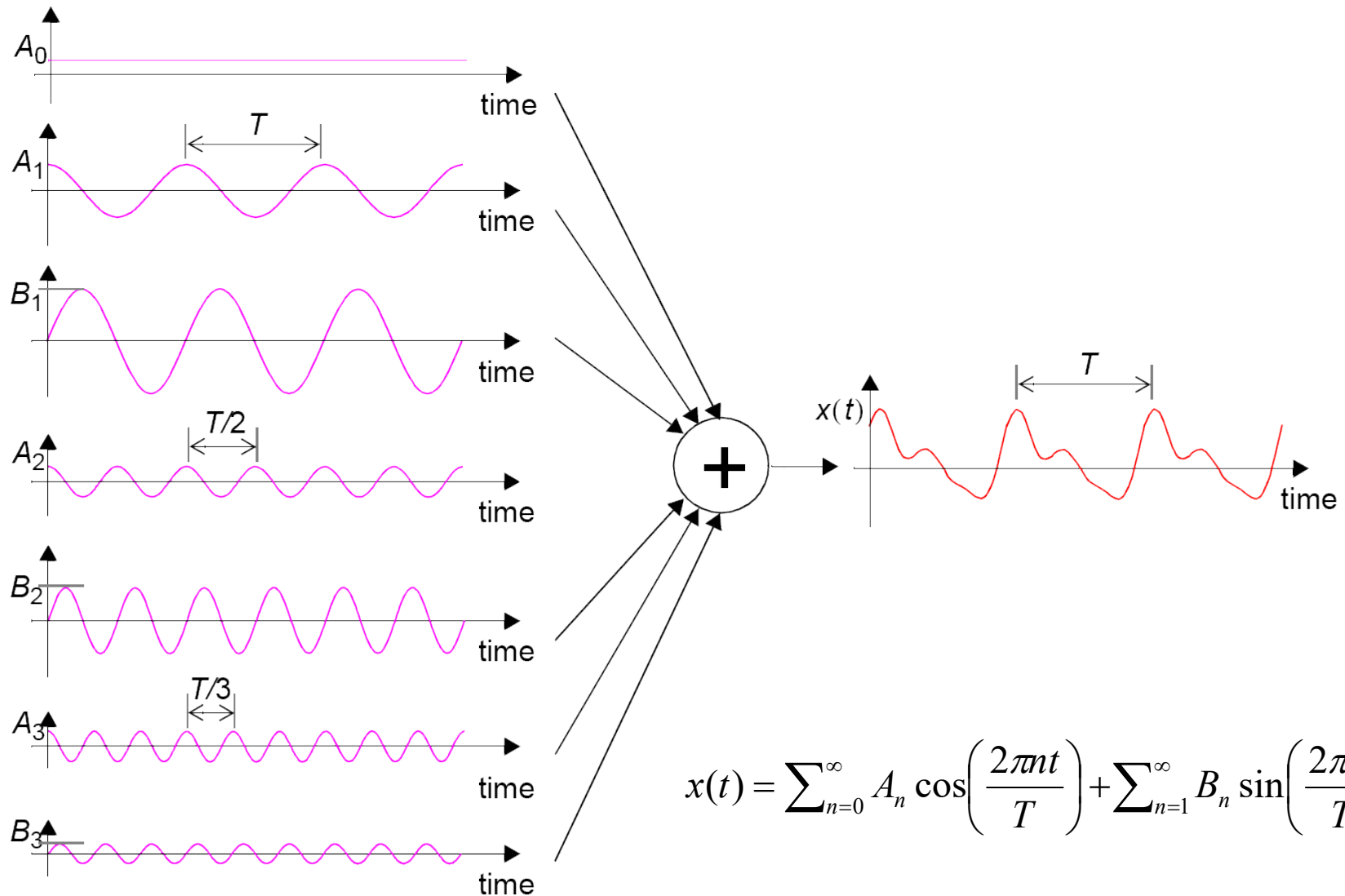
Fourier Series

Fourier Series

- Joseph Fourier (1768-1830) proposed that any periodic function can be decomposed into a sum of simple oscillating functions, namely sines and cosines.



“Any periodic function ...”



$$x(t) = \sum_{n=0}^{\infty} A_n \cos\left(\frac{2\pi n t}{T}\right) + \sum_{n=1}^{\infty} B_n \sin\left(\frac{2\pi n t}{T}\right)$$

“...can be decomposed into a sum of sines and cosines”

Fourier Series

$$f(t) = \sum_{n=0}^{\infty} A_n \cos\left(\frac{2\pi nt}{T}\right) + \sum_{n=1}^{\infty} B_n \sin\left(\frac{2\pi nt}{T}\right)$$

- Suppose a **2π periodic function** $f(t)$ integrable on $[-\pi, \pi]$
- The corresponding Fourier coefficients are:

$$A_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos(nt) dt, \quad n \geq 0$$

$$B_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \sin(nt) dt, \quad n \geq 1$$

Complex Numbers Review

Rectangular form : $C_n = \text{Re}(C_n) + i \text{Im}(C_n)$

Euler's formula : $e^{i\varphi} = \cos \varphi + i \sin \varphi$

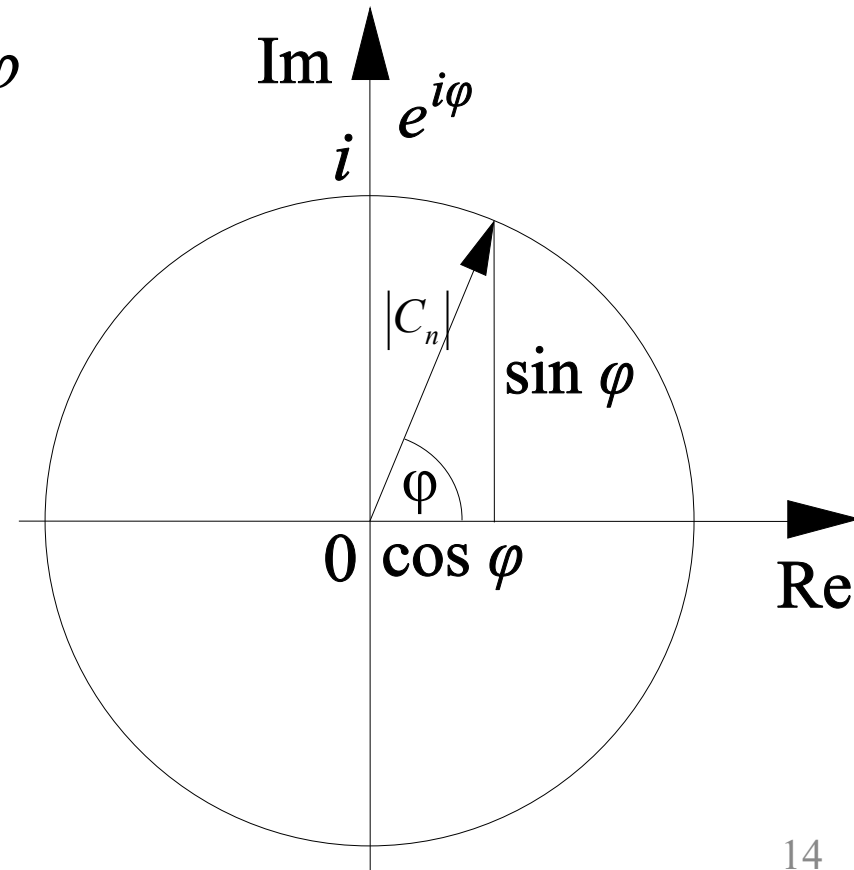
Polar form : $C_n = |C_n| e^{i\varphi}$

Magnitude : $|C_n|$

Phase : φ

$\text{Re}(C_n) = |C_n| \cos \varphi$

$\text{Im}(C_n) = |C_n| \sin \varphi$



From Real to Complex Coefficients

Real Fourier coefficients:

$$f(t) = A_0 + \sum_{n=1}^{\infty} \left[A_n \cos\left(\frac{2\pi nt}{T}\right) + B_n \sin\left(\frac{2\pi nt}{T}\right) \right]; A_n, B_n \in \mathbb{R}$$

Using the Euler relationships:

$$e^{i\omega} = \cos \omega + i \sin \omega$$

$$e^{-i\omega} = \cos(-\omega) + i \sin(-\omega) = \cos \omega - i \sin \omega$$

we can express *sin* and *cos* using exponential functions

$$\Rightarrow \cos \omega = \frac{e^{i\omega} + e^{-i\omega}}{2}; \sin \omega = \frac{e^{i\omega} - e^{-i\omega}}{2i}$$

And substituting *sin* and *cos* we get ($\omega_0 = 2\pi/T$):

$$f(t) = A_0 + \sum_{n=1}^{\infty} \left[A_n \left(\frac{e^{in\omega_0 t} + e^{-in\omega_0 t}}{2} \right) + B_n \left(\frac{e^{in\omega_0 t} - e^{-in\omega_0 t}}{2i} \right) \right]$$

and then

$$f(t) = A_0 + \sum_{n=1}^{\infty} \left(\frac{A_n - iB_n}{2} \right) e^{in\omega_0 t} + \sum_{n=-\infty}^{-1} \left(\frac{A_n + iB_n}{2} \right) e^{in\omega_0 t}$$

We can now express our Fourier series with *complex* coefficients

$$f(t) = \sum_{n=-\infty}^{\infty} C_n e^{in\omega_0 t}; C_n \in \mathbb{C}$$

Complex Fourier Coefficients

For 2π periodic function:

Fourier series

$$f(t) = \sum_{n=-\infty}^{\infty} C_n e^{in\omega_0 t}$$

Fourier coefficients

$$C_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) e^{-in\omega_0 t} dt$$

Rem (s. 14): $C_n = |C_n| e^{i\varphi}$ Magnitude: $|C_n|$ Phase: φ

Fourier Series for Arbitrary Period

$$f(t) = \sum_{n=0}^{\infty} A_n \cos\left(\frac{2\pi nt}{T}\right) + \sum_{n=1}^{\infty} B_n \sin\left(\frac{2\pi nt}{T}\right)$$

- Suppose $f(t)$ **periodic in T** and integrable on this period
- The corresponding Fourier coefficients are:

$$A_n = \frac{2}{T} \int_{-T/2}^{T/2} f(t) \cos\left(\frac{2\pi nt}{T}\right) dt, \quad n \geq 0$$

$$B_n = \frac{2}{T} \int_{-T/2}^{T/2} f(t) \sin\left(\frac{2\pi nt}{T}\right) dt, \quad n \geq 1$$

Note: compare with s. 13 expressions with $T=2\pi$

Complex Coefficient for Arbitrary Period

Fourier series $f(t) = \sum_{n=-\infty}^{\infty} C_n e^{in\omega_0 t}$

Fourier coefficients
for T periodic function $C_n = \frac{1}{T} \int_{-T/2}^{T/2} f(t) e^{-in\omega_0 t} dt$

Note: compare with s. 16 expressions with $T=2\pi$

Rem (s. 14): $C_n = |C_n| e^{i\varphi}$ Magnitude: $|C_n|$ Phase: φ 18

Examples

Matlab demo on Fourier series

Associated to this book:

J. H. McClellan, R. W. Schafer, M. A. Yoder

“DSP First: A Multimedia Approach”, Prentice Hall, 1999.

Available for download here:

<https://dspfirst.gatech.edu/matlab/>

Fourier Transform

From Fourier Series to Transform

- $f(t)$
 - an aperiodic signal
 - view it as the limit of a periodic signal as $T \rightarrow \infty$
 - is a piecewise continuous, integrable (function space L^1) or even a square-integrable (function space L^2) function

$$\int_{-\infty}^{\infty} |f(x)| dx < \infty \qquad \int_{-\infty}^{\infty} |f(x)|^2 dx < \infty$$

- For a periodic signal, the harmonic components are spaced $\omega_0 = 2\pi/T$ apart ($\omega_0 =$ fundamental angular frequency)
- As $T \rightarrow \infty$, $\omega_0 \rightarrow 0$, and harmonic components are spaced closer and closer in frequency forming a continuous spectrum

Fourier Transform

Unitary, ordinary frequency notation

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} f(t) \cdot e^{-i2\pi\xi t} dt$$

**Fourier
Transform**

$$f(t) = \int_{-\infty}^{\infty} \hat{f}(\xi) \cdot e^{i2\pi t\xi} d\xi$$

**Inverse Fourier
Transform**

Notes:

- \hat{f} is often replaced with F
- the direct transform is also called “analysis equation”
- the inverse transform is also called “synthesis equation”

Fourier Transform

Non-unitary, angular frequency notation

$$F(\omega) = \int_{-\infty}^{+\infty} f(t) e^{-i\omega t} dt$$

**Fourier
Transform**

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} F(\omega) e^{i\omega t} d\omega$$

**Inverse Fourier
Transform**

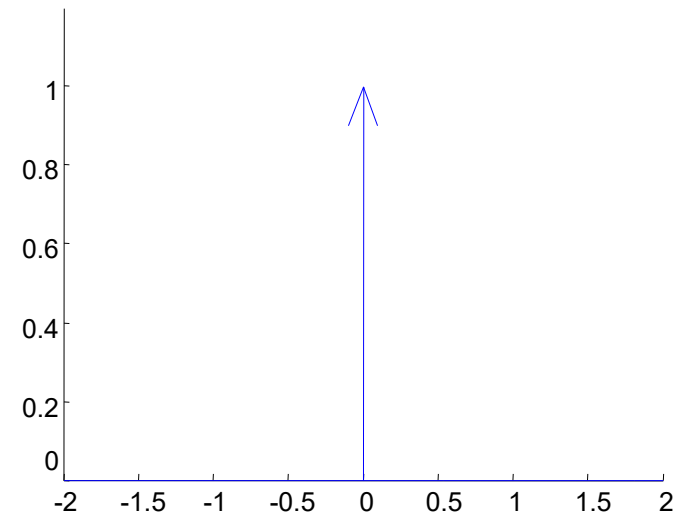
Notes:

- $\omega = 2\pi\xi \rightarrow$ obtained from unitary, ordinary frequency transform with $\xi = \omega/2\pi$
- F can be replaced with f^\wedge
- In electrical engineering i is substituted by j (“i” booked for current)
- Often, in order to emphasize the frequency response aspect, the imaginary aspect of the transform is emphasized: $F(\omega)$, $F(i\omega)$, or $F(j\omega)$ are all equivalent notations

Dirac Delta Function

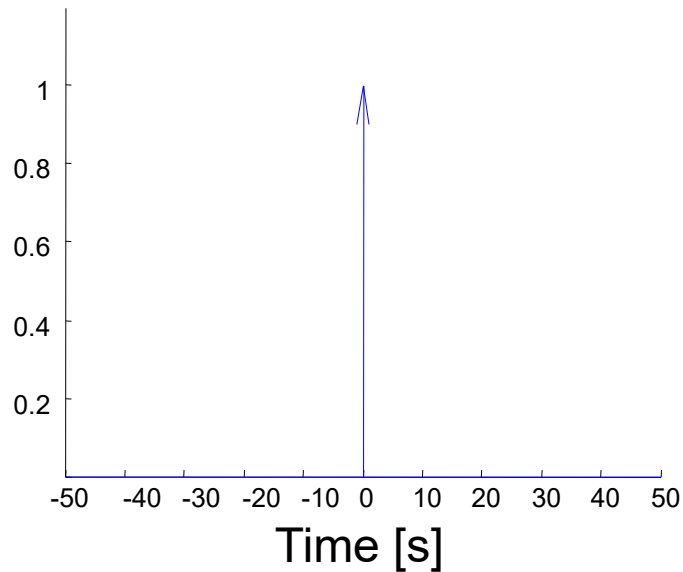
$$\delta(t) = \begin{cases} +\infty, & t = 0 \\ 0, & t \neq 0 \end{cases}$$

$$\int_{-\infty}^{+\infty} \delta(t) dt = 1$$



Note: The height of the arrow (or small circle) is usually used to specify the value of any multiplicative constant, which will give the area under the function; the other convention is to write the area next to the arrowhead. [Wikipedia]

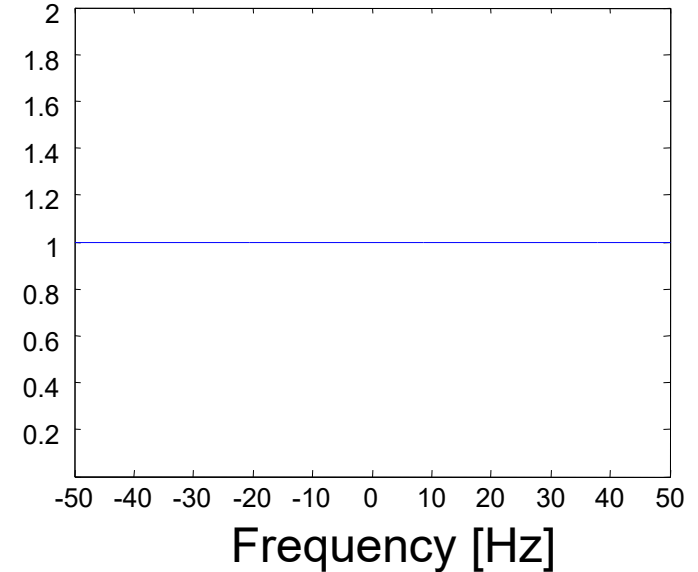
Common Fourier Transform



$$f(t) = \delta(t)$$

FT
(analysis equation)

IFT
(synthesis equation)



$$\hat{f}(\xi) = 1$$

FT of a Dirac Impulse

$$f(t) = \delta(t)$$

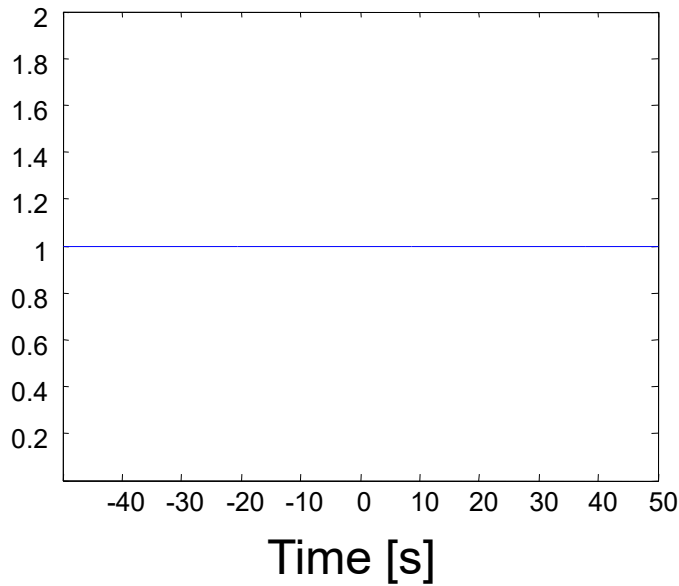
$$\hat{f}(\xi) = \int_{-\infty}^{+\infty} f(t) e^{-i2\pi\xi t} dt = \int_{-\infty}^{+\infty} \delta(t) e^{-i2\pi\xi t} dt = 1$$

Analysis
equation

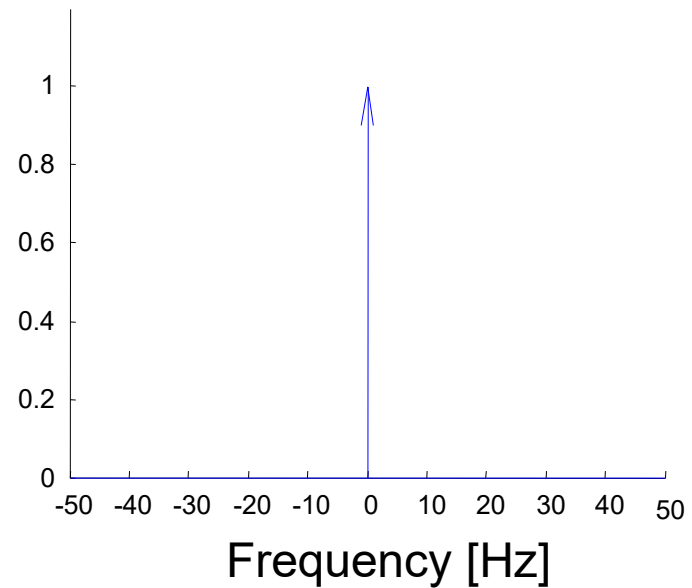
$$f(t) = \int_{-\infty}^{+\infty} \hat{f}(\xi) e^{i2\pi t \xi} d\xi = \int_{-\infty}^{+\infty} e^{i2\pi t \xi} d\xi = \delta(t)$$

Synthesis
equation

Common Fourier Transform

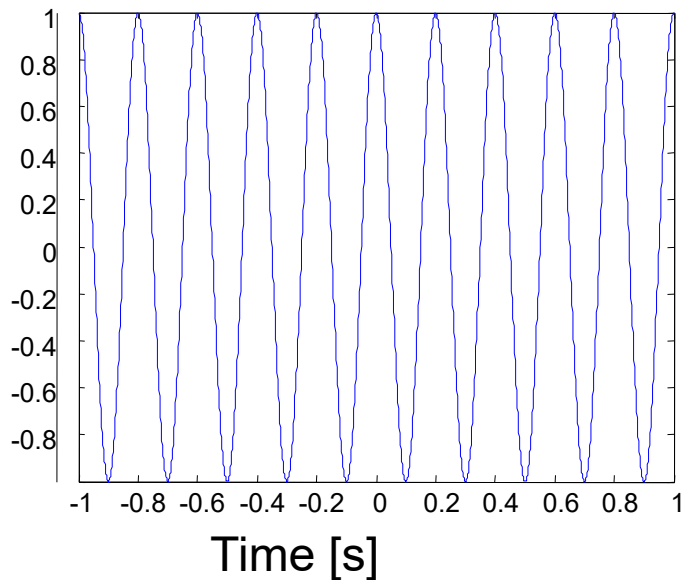


$$f(t) = 1$$

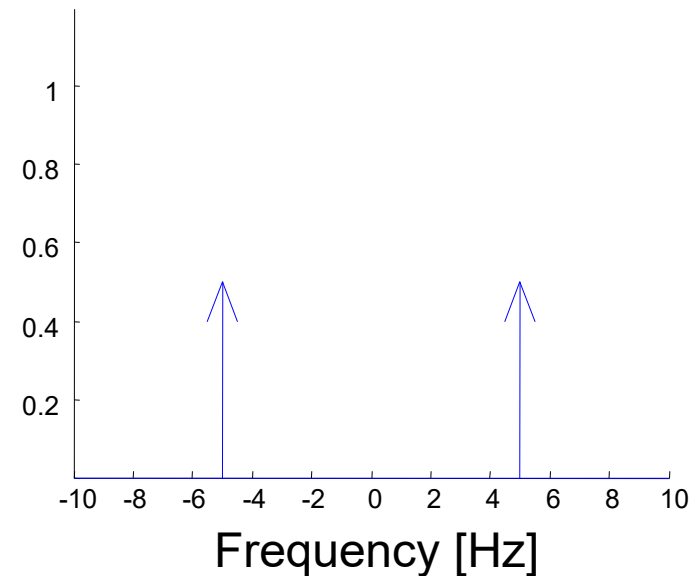


$$\hat{f}(\xi) = \delta(\xi)$$

Common Fourier Transform

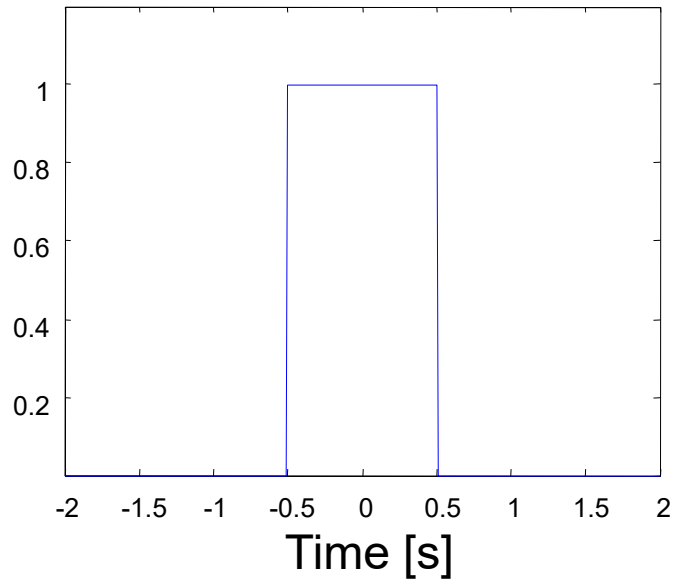


$$f(t) = \cos(2\pi at)$$

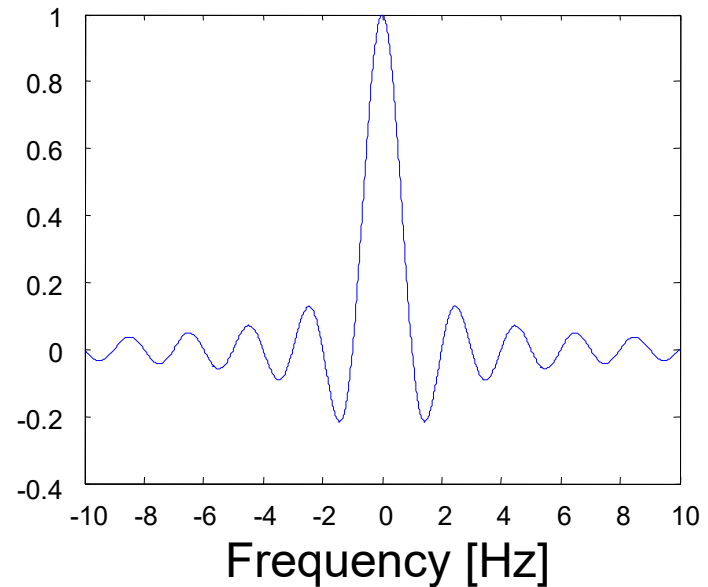


$$\hat{f}(\xi) = \frac{\delta(\xi - a) + \delta(\xi + a)}{2}$$

Common Fourier Transform



$$f(t) = \text{rect}(t)$$



$$\hat{f}(\xi) = \text{sinc}(\xi)$$

Properties

Linearity

$$h(t) = af(t) + bg(t) \Rightarrow \hat{h}(\xi) = a\hat{f}(\xi) + b\hat{g}(\xi)$$

Translation

$$h(t) = f(t - t_0) \Rightarrow \hat{h}(\xi) = e^{-2\pi i t_0 \xi} \hat{f}(\xi)$$

Modulation

$$h(t) = e^{2\pi i t \xi_0} f(t) \Rightarrow \hat{h}(\xi) = \hat{f}(\xi - \xi_0)$$

Scaling

$$h(t) = f(at) \Rightarrow \hat{h}(\xi) = \frac{1}{|a|} \hat{f}\left(\frac{\xi}{a}\right)$$

Convolution

$$h(t) = (f * g)(t) \Rightarrow \hat{h}(\xi) = \hat{f}(\xi) \cdot \hat{g}(\xi)$$

See s. 40 onwards

Fourier Tables

Fourier Transforms Table

Note $\nu = \omega$
on s. 21

Note $x = t$
on s. 20/21

Simple
example
s. 34

	Function	Fourier transform unitary, ordinary frequency	Fourier transform unitary, angular frequency	Fourier transform non-unitary, angular frequency
	$f(x)$	$\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x)e^{-2\pi i x \xi} dx$	$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)e^{-i\omega x} dx$	$\hat{f}(\nu) = \int_{-\infty}^{\infty} f(x)e^{-i\nu x} dx$
201	$\text{rect}(ax)$	$\frac{1}{ a } \cdot \text{sinc}\left(\frac{\xi}{a}\right)$	$\frac{1}{\sqrt{2\pi a^2}} \cdot \text{sinc}\left(\frac{\omega}{2\pi a}\right)$	$\frac{1}{ a } \cdot \text{sinc}\left(\frac{\nu}{2\pi a}\right)$
202	$\text{sinc}(ax)$	$\frac{1}{ a } \cdot \text{rect}\left(\frac{\xi}{a}\right)$	$\frac{1}{\sqrt{2\pi a^2}} \cdot \text{rect}\left(\frac{\omega}{2\pi a}\right)$	$\frac{1}{ a } \cdot \text{rect}\left(\frac{\nu}{2\pi a}\right)$
203	$\text{sinc}^2(ax)$	$\frac{1}{ a } \cdot \text{tri}\left(\frac{\xi}{a}\right)$	$\frac{1}{\sqrt{2\pi a^2}} \cdot \text{tri}\left(\frac{\omega}{2\pi a}\right)$	$\frac{1}{ a } \cdot \text{tri}\left(\frac{\nu}{2\pi a}\right)$
204	$\text{tri}(ax)$	$\frac{1}{ a } \cdot \text{sinc}^2\left(\frac{\xi}{a}\right)$	$\frac{1}{\sqrt{2\pi a^2}} \cdot \text{sinc}^2\left(\frac{\omega}{2\pi a}\right)$	$\frac{1}{ a } \cdot \text{sinc}^2\left(\frac{\nu}{2\pi a}\right)$
205	$e^{-ax}u(x)$	$\frac{1}{a + 2\pi i \xi}$	$\frac{1}{\sqrt{2\pi}(a + i\omega)}$	$\frac{1}{a + i\nu}$
206	$e^{-\alpha x^2}$	$\sqrt{\frac{\pi}{\alpha}} \cdot e^{-\frac{(\pi \xi)^2}{\alpha}}$	$\frac{1}{\sqrt{2\alpha}} \cdot e^{-\frac{\omega^2}{4\alpha}}$	$\sqrt{\frac{\pi}{\alpha}} \cdot e^{-\frac{\nu^2}{4\alpha}}$
207	$e^{-a x }$	$\frac{2a}{a^2 + 4\pi^2 \xi^2}$	$\sqrt{\frac{2}{\pi}} \cdot \frac{a}{a^2 + \omega^2}$	$\frac{2a}{a^2 + \nu^2}$

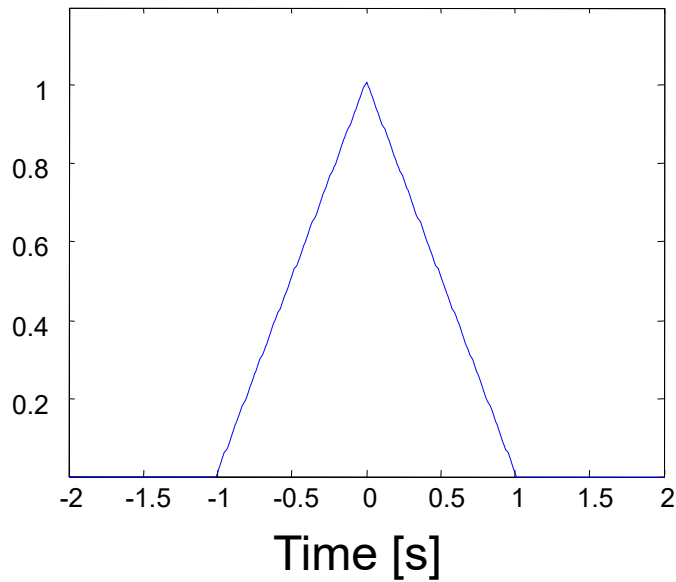
Fourier Properties Table

	Function	Fourier transform unitary, ordinary frequency	Fourier transform unitary, angular frequency	Fourier transform non-unitary, angular frequency
	$f(x)$	$\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x)e^{-2\pi i x \xi} dx$	$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)e^{-i\omega x} dx$	$\hat{f}(\nu) = \int_{-\infty}^{\infty} f(x)e^{-i\nu x} dx$
101	$a \cdot f(x) + b \cdot g(x)$	$a \cdot \hat{f}(\xi) + b \cdot \hat{g}(\xi)$	$a \cdot \hat{f}(\omega) + b \cdot \hat{g}(\omega)$	$a \cdot \hat{f}(\nu) + b \cdot \hat{g}(\nu)$
102	$f(x - a)$	$e^{-2\pi i a \xi} \hat{f}(\xi)$	$e^{-i a \omega} \hat{f}(\omega)$	$e^{-i a \nu} \hat{f}(\nu)$
103	$e^{2\pi i a x} f(x)$	$\hat{f}(\xi - a)$	$\hat{f}(\omega - 2\pi a)$	$\hat{f}(\nu - 2\pi a)$
104	$f(ax)$	$\frac{1}{ a } \hat{f}\left(\frac{\xi}{a}\right)$	$\frac{1}{ a } \hat{f}\left(\frac{\omega}{a}\right)$	$\frac{1}{ a } \hat{f}\left(\frac{\nu}{a}\right)$
105	$\hat{f}(x)$	$f(-\xi)$	$f(-\omega)$	$2\pi f(-\nu)$
106	$\frac{d^n f(x)}{dx^n}$	$(2\pi i \xi)^n \hat{f}(\xi)$	$(i\omega)^n \hat{f}(\omega)$	$(i\nu)^n \hat{f}(\nu)$
107	$x^n f(x)$	$\left(\frac{i}{2\pi}\right)^n \frac{d^n \hat{f}(\xi)}{d\xi^n}$	$i^n \frac{d^n \hat{f}(\omega)}{d\omega^n}$	$i^n \frac{d^n \hat{f}(\nu)}{d\nu^n}$
108	$(f * g)(x)$	$\hat{f}(\xi)\hat{g}(\xi)$	$\sqrt{2\pi} \hat{f}(\omega)\hat{g}(\omega)$	$\hat{f}(\nu)\hat{g}(\nu)$
109	$f(x)g(x)$	$(\hat{f} * \hat{g})(\xi)$	$\frac{(\hat{f} * \hat{g})(\omega)}{\sqrt{2\pi}}$	$\frac{1}{2\pi} (\hat{f} * \hat{g})(\nu)$

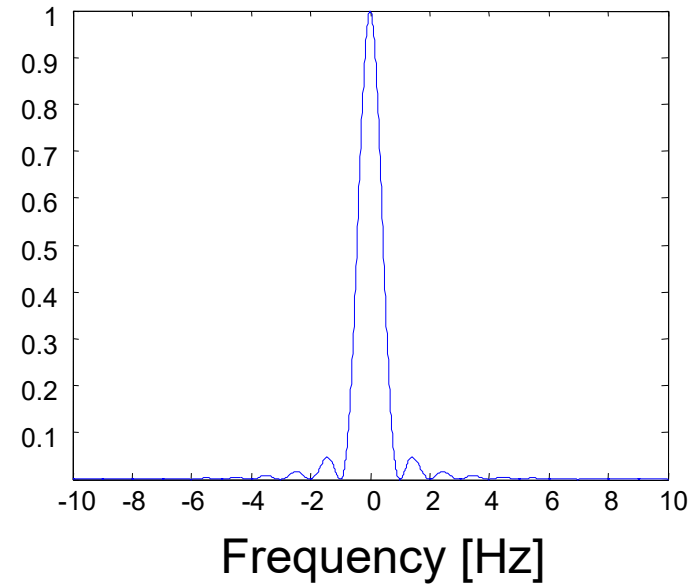
Combined example s. 35



Simple Example

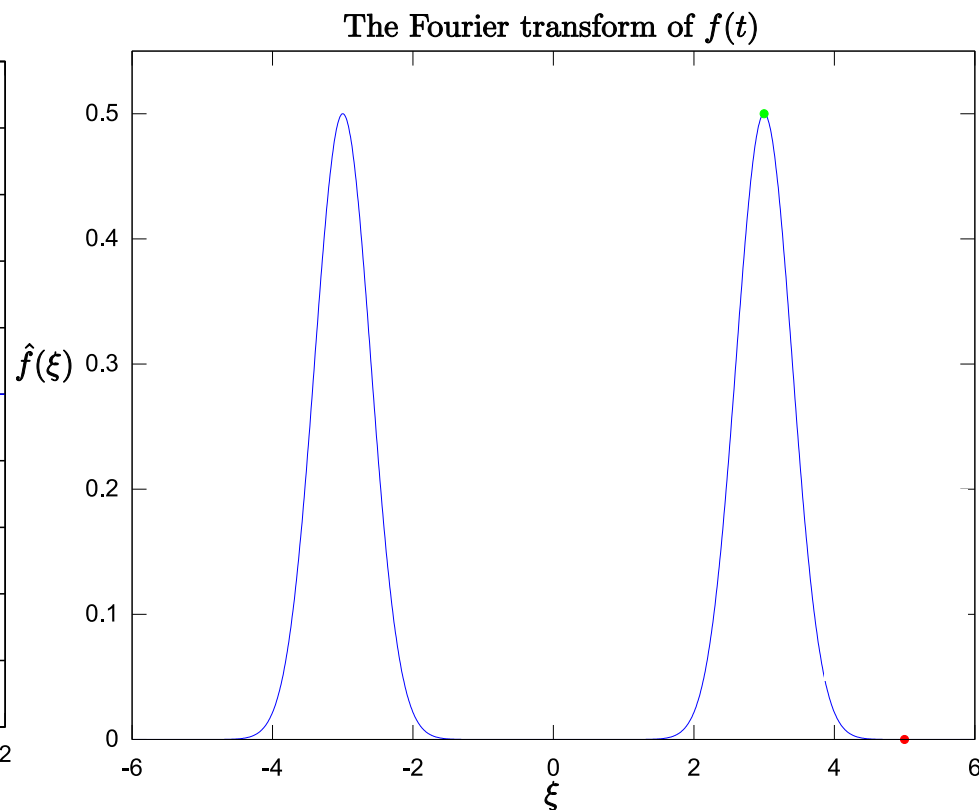
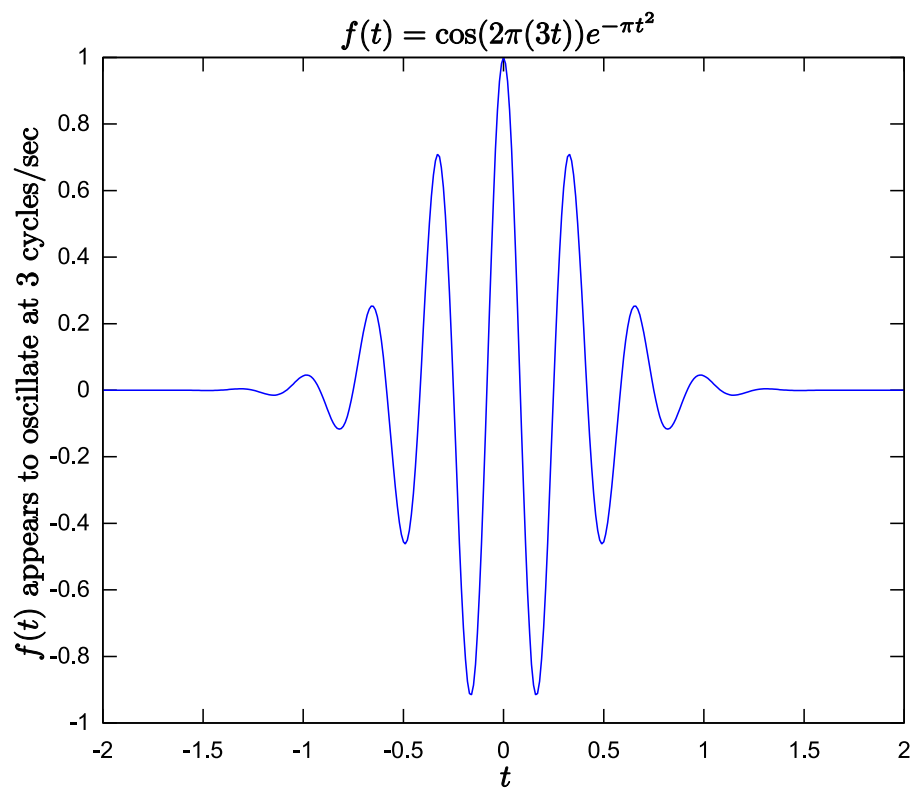


$$f(t) = \text{trian}(t)$$



$$\hat{f}(\xi) = \text{sinc}^2(\xi)$$

Combined Example



Discrete Fourier Transform

Discrete Fourier Transform

$$\hat{f}(\xi) = F(\xi) = \int_{-\infty}^{\infty} f(t) \cdot e^{-2\pi i \xi t} dt$$

$$X[k] = \sum_{n=0}^{N-1} x[n] \cdot e^{\frac{-2\pi i}{N} kn}, k = 0, \dots, N-1$$

- Input and output sequences are both **finite** (the integral becomes a finite sum)
- Both time and frequency domain are discrete (note the “[]”)
- Both time and frequency domain are defined on the positive axis
- Matlab, operating on a computer (i.e. a digital device) can only emulate continuity/infinity and therefore use this discrete version with an adjustable discretization level (in time, frequency, and amplitude) and finite bounds
- Also, since Matlab index start from 1, formulation is typically $x[n+1]$, $X[k+1]$

Discrete Fourier Transform

$$X[k] = \sum_{n=0}^{N-1} x[n] \cdot e^{-\frac{2\pi i}{N}kn}, k = 0, \dots, N-1 \quad \text{DFT}$$

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] \cdot e^{\frac{2\pi i}{N}kn}, k = 0, \dots, N-1 \quad \text{Inverse DFT}$$

- Different from the Discrete-Time Fourier Transform (DTFT), see Week 4
- Efficient implementation, for instance in Matlab, is done using a **Fast Fourier Transform (FFT)** algorithm (e.g., Cooley and Tukey, 1965)
- Clear correspondence with the Fourier series (see slide 16)

Differences among Fourier Series, Continuous and Discrete Transforms

FT $\hat{f}(\xi) = F(\xi) = \int_{-\infty}^{\infty} f(t) \cdot e^{-2\pi i \xi t} dt$

DFT $X[k] = \sum_{n=0}^{N-1} x[n] \cdot e^{-\frac{2\pi i}{N} kn}, k = 0, \dots, N-1$

FS $f(t) = \sum_{n=-\infty}^{\infty} C_n e^{in\omega_0 t} \quad C_n = \frac{1}{T} \int_{-T/2}^{T/2} f(t) e^{-in\omega_0 t} dt$

Convolution

Convolution

$$(f * g)(t) = \int_{-\infty}^{\infty} f(\tau) \cdot g(t - \tau) d\tau$$

- For each value of t :

1. Flip (reflect) g

$$1) g(\tau) \rightarrow g(-\tau)$$

2. Shift g by t

$$2) g(-\tau) \rightarrow g(t - \tau)$$

3. Multiply f and g

$$3) f(\tau) \cdot g(t - \tau)$$

4. Integrate over τ

$$4) \int_{-\infty}^{\infty} f(\tau) \cdot g(t - \tau) d\tau$$

- Note that the result does **not** depend on τ !

Examples

Matlab demo on Continuous Convolution

Associated to this book:

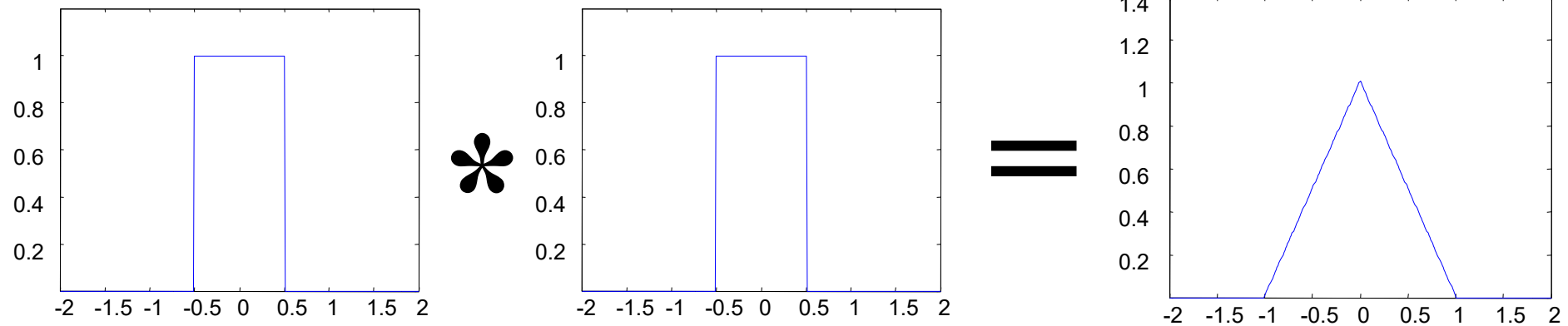
J. H. McClellan, R. W. Schafer, M. A. Yoder

“DSP First: A Multimedia Approach”, Prentice Hall, 1999.

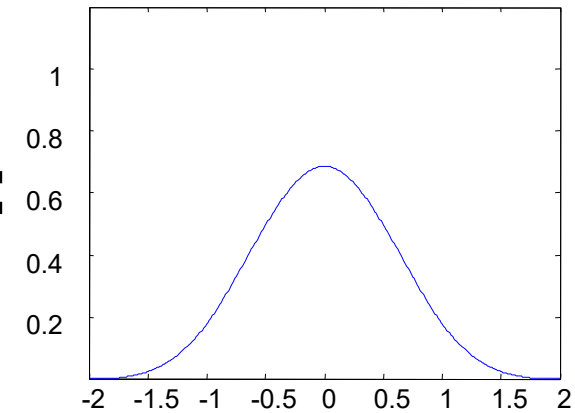
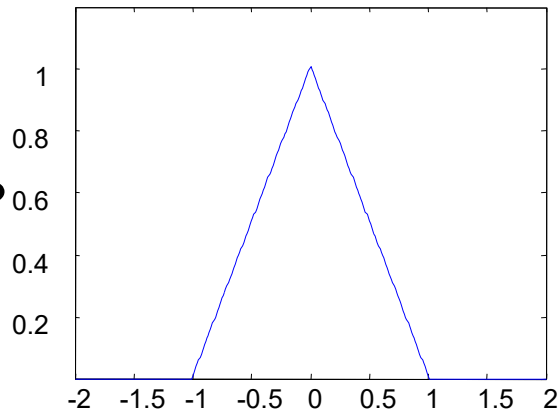
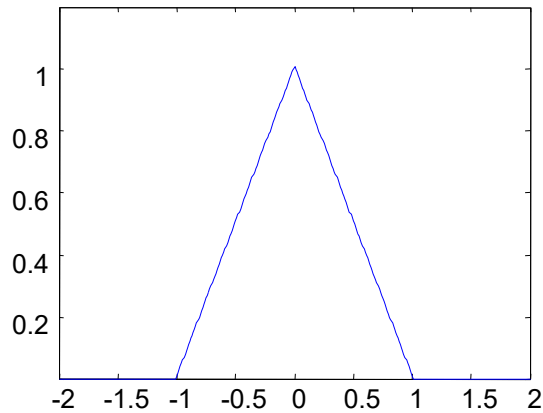
Available for download here:

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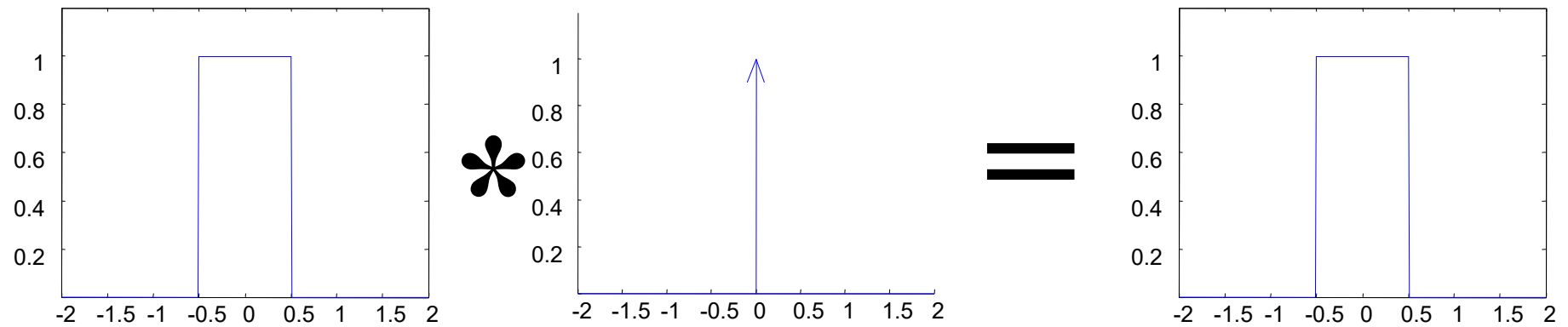
Examples



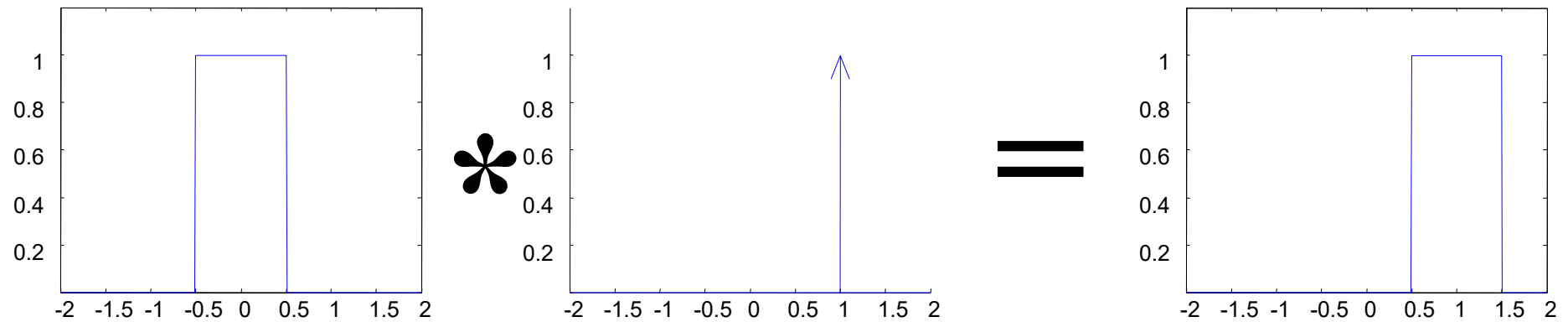
Examples



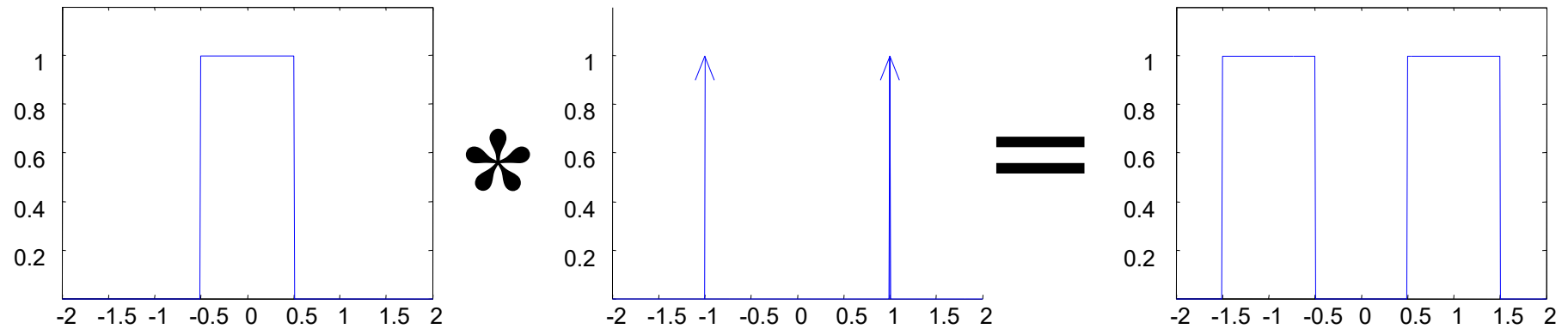
Examples



Examples



Examples



Convolution in Time and Frequency Domains

Time domain

Frequency domain

$$h(t) = (f * g)(t) \Leftrightarrow \hat{h}(\xi) = \hat{f}(\xi) \cdot \hat{g}(\xi)$$

$$h(t) = f(t) \cdot g(t) \Leftrightarrow \hat{h}(\xi) = (\hat{f} * \hat{g})(\xi)$$

$$h(t) = (f * g)(t) \Leftrightarrow H(\omega) = F(\omega) \cdot G(\omega)$$

$$h(t) = f(t) \cdot g(t) \Leftrightarrow H(\omega) = \frac{1}{2\pi} (F * G)(\omega)$$

Unitary, ordinary
frequency

Non-unitary, angular
frequency

Convolution Properties

Commutativity

$$f * g = g * f$$

Associativity

$$f * (g * h) = (f * g) * h$$

Distributivity

$$f * (g + h) = (f * g) + (f * h)$$

Associativity with scalar multiplication

$$a(f * g) = (af) * g = f * (ag)$$

Discrete Convolution

$$(f * g)(t) = \int_{-\infty}^{\infty} f(\tau) \cdot g(t - \tau) d\tau$$



$$(f * g)[n] = \sum_{m=-\infty}^{\infty} f[m] \cdot g[n - m]$$

Notes:

- Similar to the continuous version
- The integral becomes an infinite sum (note: **in theory** no bounds on the sum by definition as in the DFT)
- **In practice**, a computer (i.e. a digital device), running Matlab for instance, can only emulate continuity and therefore discrete time and quantized amplitude as well as finite bounds of the convolution window are used

Examples

Matlab demo on Discrete Convolution

Associated to this book:

J. H. McClellan, R. W. Schafer, M. A. Yoder

“DSP First: A Multimedia Approach”, Prentice Hall, 1999.

Available for download here:

<https://dspfirst.gatech.edu/matlab/>

Conclusion

Take Home Messages

- Fourier decomposition: every periodic signal can be decomposed into a sum of sines and cosines
- Fourier coefficients are the weights in this sum
- Fourier Transform (FT) for aperiodic signals
- Implementation on computers of Fourier transform is discretized and bound: Discrete Fourier Transform (DFT)
- The Fast FT (FFT) is a fast algorithm for solving the DFT
- Properties of FT and tables help to solve analytically analysis and synthesis equations
- Multiplication in time domain means convolution in frequency domain and vice versa
- Continuous vs. discrete convolution (analogy: FT vs. DFT)

Additional Literature – Week 2

Related course material

- Information, Calcul, Communication (ICC)
- Analysis IV

Books

- J. H. McClellan, R. W. Schafer, M. A. Yoder
“DSP First: A Multimedia Approach”, Prentice Hall, 1999.
- A. Oppenheim and A. S. Willsky with S. Nawab, “Signals and Systems”, Prentice Hall, 1997.