EPFL

Disal

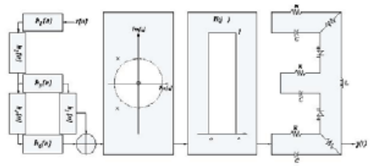
Signals, Instruments, and Systems – W4

Introduction to Signal Processing – System Properties, Responses, Functions, More Transforms, and Filters

Outline

- System properties
- More transforms
- Frequency responses and transfer functions
- Motivating examples for filters
- Basic filters
- Bode plots
- Analog and digital filters
- Filter order and type

Acknowledgment for Selected Slides



Signals and Systems

Fall 2003

Lecture #1

Prof. Alan S. Willsky

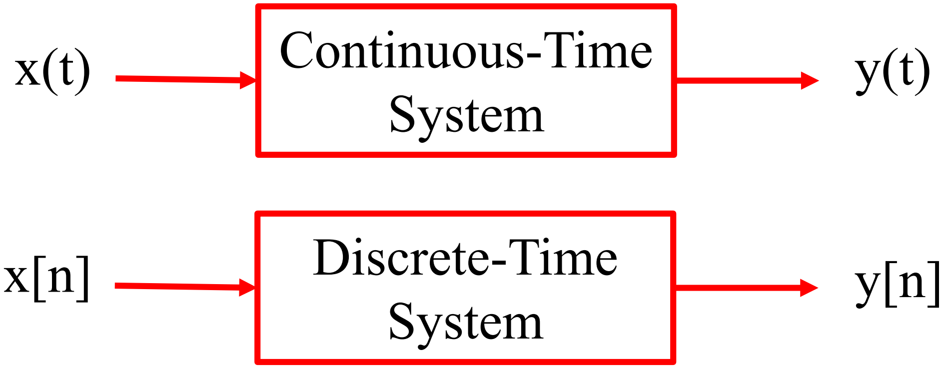
4 September 2003

1) Administrative details
2) Signals
3) Systems
4) For examples ...

1

System Properties

System Properties



- Knowing what system properties are fulfilled helps the analysis of the system and has practical implications
- Three key properties: causality, time-invariance, and linearity

Causality

CAUSALITY

- A system is causal if the output does not anticipate future values of the input, i.e., if the output at any time depends only on values of the input up to that time.

- All real-time physical systems are causal, because time only moves forward. Effect occurs after cause. (Imagine if you own a noncausal system whose output depends on tomorrow's stock price.)

- Causality does not apply to spatially varying signals. (We can move both left and right, up and down.)

- Causality does not apply to systems processing recorded signals, e.g. taped sports games vs. live broadcast.

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Causality

CAUSALITY (continued)

- Mathematically (in CT): A system $x(t) \rightarrow y(t)$ is causal if

when	$x_1(t) \rightarrow y_1(t)$ $\quad x_2(t) \rightarrow y_2(t)$	
and	$x_1(t) = x_2(t)$	for all $t \leq t_o$
Then	$y_1(t) = y_2(t)$	for all $t \leq t_o$

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Time-Invariance

TIME-INVARIANCE (TI)

Informally, a system is time-invariant (TI) if its behavior does not depend on what time it is.

- Mathematically (in DT): A system $x[n] \rightarrow y[n]$ is TI if for any input $x[n]$ and any time shift n_0,

  If $\quad x[n] \rightarrow y[n]$

  then $\quad x[n - n_0] \rightarrow y[n - n_0]$.

- Similarly for a CT time-invariant system,

  If $\quad x(t) \rightarrow y(t)$

  then $\quad x(t - t_o) \rightarrow y(t - t_o)$.

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Linearity

A (CT) system is linear if it has the superposition property:

If $x_1(t) \rightarrow y_1(t)$ and $x_2(t) \rightarrow y_2(t)$

then $ax_1(t) + bx_2(t) \rightarrow ay_1(t) + by_2(t)$

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Additional Definitions

- LTI: Linear Time-Invariant systems
- SISO: Single-Input Single-Output system
- MIMO: Multi-Input Multi-Output system

In this course, we will consider only SISO filters in an operational regime respecting the LTI properties

More Transforms

Motivation for More Transforms

Motivation for the Laplace Transform

- CT Fourier transform enables us to do a lot of things, e.g.
  - Analyze frequency response of LTI systems
  - Sampling
  - Modulation

  ⋮
- Why do we need yet another transform?
- One view of Laplace Transform is as an *extension* of the Fourier transform to allow analysis of broader class of signals and systems
- In particular, Fourier transform *cannot* handle large (and important) classes of signals and *unstable* systems, i.e. when

$$\int_{-\infty}^{\infty} |x(t)|dt = \infty$$

Note: compare with W2, s.21

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Laplace Transform

$$F(s) = \mathcal{L}\{f(t)\} = \int_{-\infty}^{\infty} e^{-st} f(t) dt$$

$$s = \sigma + i\omega$$

Note: the standard Laplace transform is involving an integral between 0 and ∞; the bounds - ∞ to ∞ are used for the bilateral Laplace transform which is what we will use in this course (calling by simplicity Laplace transform)

Fourier - Laplace

$$F(\omega) = F\{f(t)\} = \mathcal{L}\{f(t)\}\,|_{s=i\omega}$$

$$= F(s)\,|_{s=i\omega} = \int_{-\infty}^{\infty} e^{-i\omega t} f(t)\,dt$$

Note: non-unitary, angular frequency, see W2, s. 23

The Fourier Transform is therefore a special case of the Laplace Transform

Fourier: frequency response (in stationary conditions, especially in signal processing)
Laplace: impulse response (also in transient conditions, especially in control)

Discrete-Time Fourier Transform

- Corresponds to the Fourier Transform for discrete-time signals (different from the Discrete Fourier Transform, a finite, bounded approximation of the Fourier Transform for digital devices)
- Transform discrete-time signals from time-domain to frequency domain (continuous spectrum)

$$X(\omega) = \sum_{n=-\infty}^{\infty} x[n] \cdot e^{-i\omega n}$$

Note: compare with DFT notation on W2, s. 38:

$$X[k] = \sum_{n=0}^{N-1} x[n] \cdot e^{\frac{-2\pi i}{N}kn}$$

$$k = 0, \cdots, N-1$$

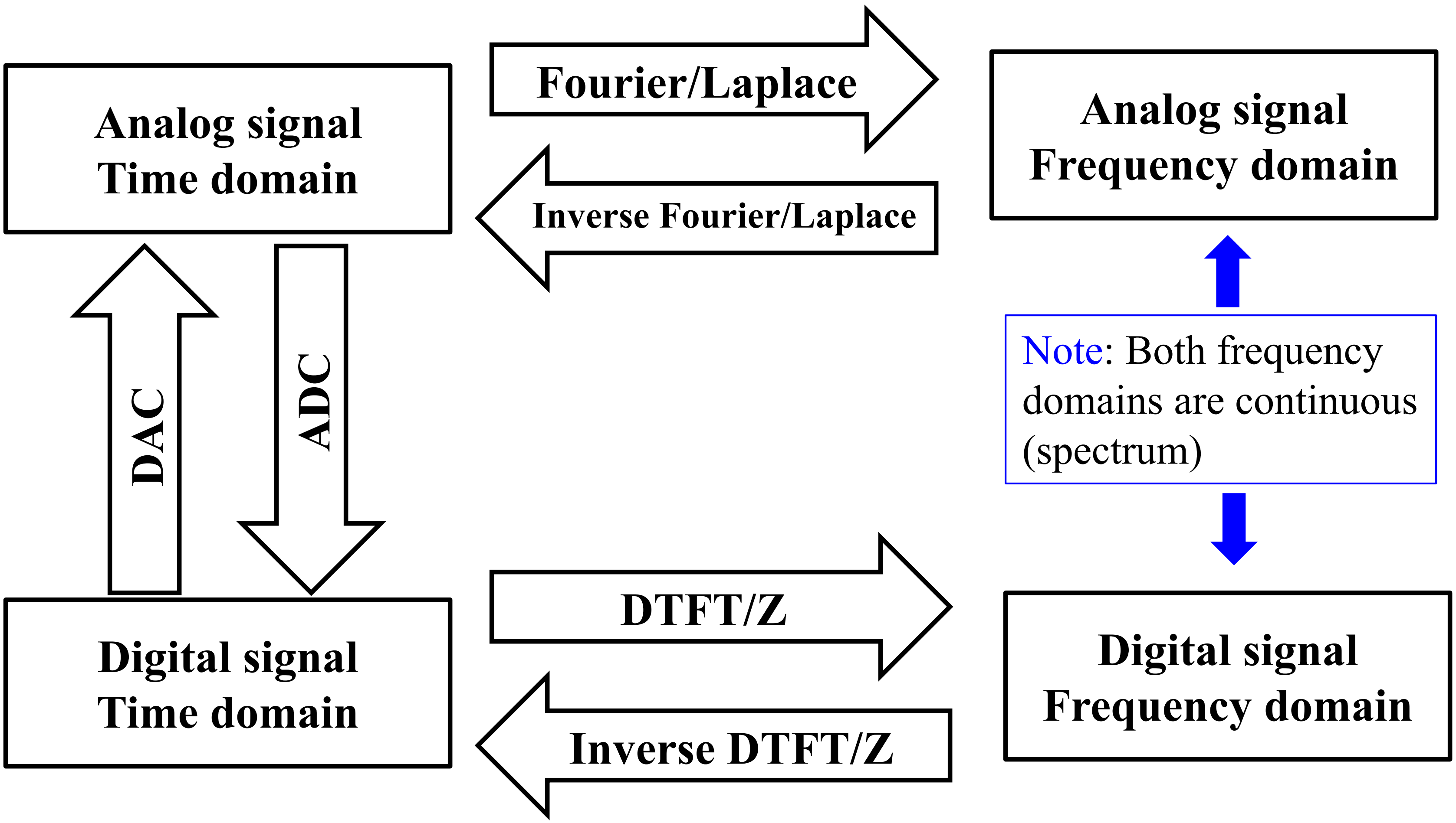
Z-Transform

- Corresponds to Laplace transform for time-discrete signals
- Transform signals from time-domain to frequency domain
- The Discrete-Time Fourier Transform is a special case of the Z-Transform with $z = e^{i\omega}$ (see s. 15)

$$X(z) = Z\{x[n]\} = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$

$z = Ae^{i\phi}$ **or** $z = A(\cos\phi + \boldsymbol{i}\sin\phi)$

Transform Overview



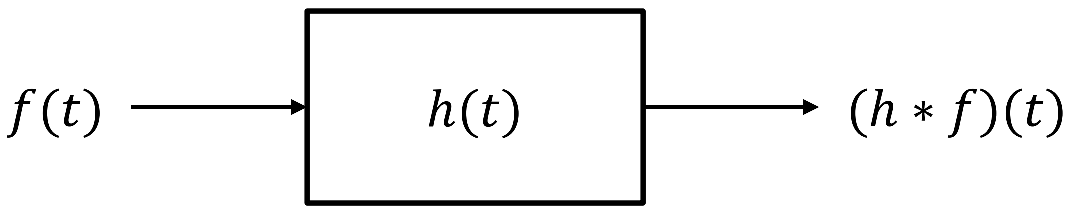
Frequency Responses and Transfer Functions

Frequency Response

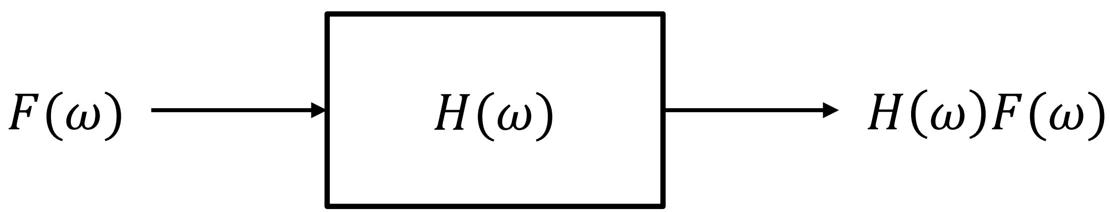
h(t) : impulse response
H(ω) : frequency response (stationary regime) of the filter/system, i.e Fourier transform of h(t)

Note: impulse response = response to an impulse
$f(t) = \delta(\mathrm{t})$
$(h * f)(t) = h(t)$
see s. 45, W2

Time domain

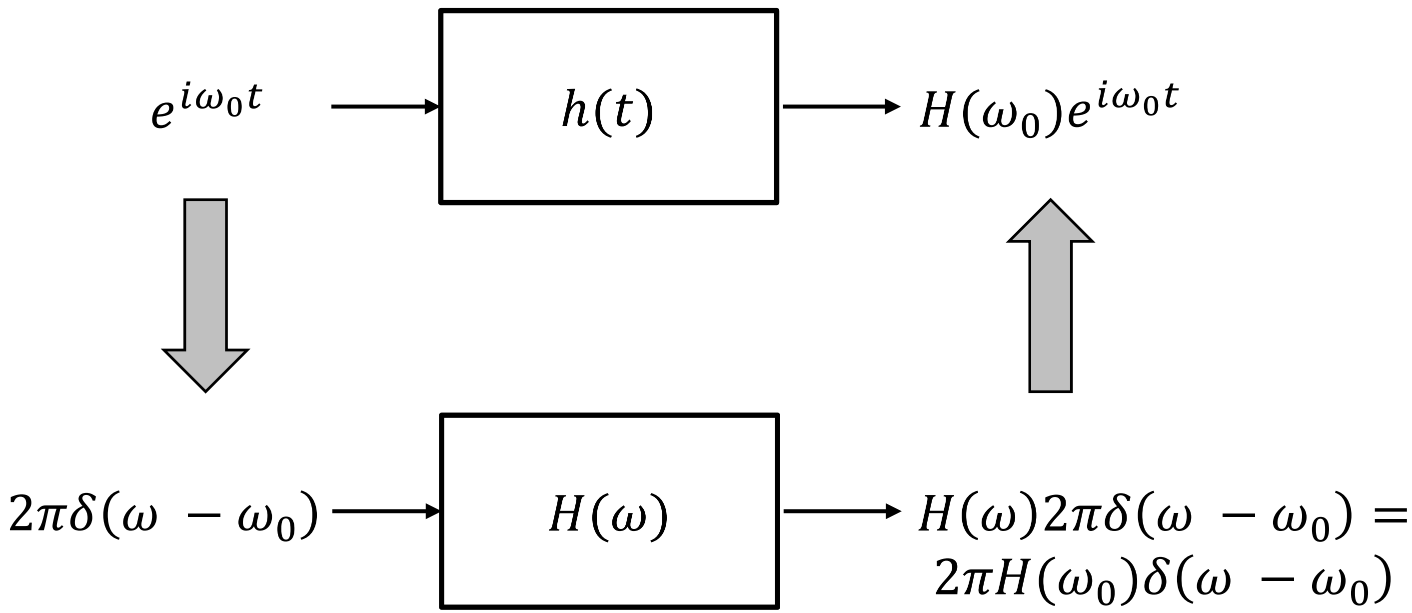


Frequency domain



Reminder: a convolution in the time domain is a multiplication in the frequency domain

Frequency Response



A single frequency comes out of a LTI filter multiplied with the value of the filter at that frequency.

Frequency Response

Continuous time

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{ik\omega_0 t} \longrightarrow \boxed{h(t)} \longrightarrow y(t) = \sum_{k=-\infty}^{\infty} H(k\omega_0) a_k e^{ik\omega_0 t}$$

$$a_k \rightarrow \underbrace{H(k\omega_0)}_{\textbf{Gain}} a_k$$

$$H(k\omega_0) = \underbrace{|H(k\omega_0)|}_{\textbf{Amplitude}} e^{\overbrace{i\measuredangle H(k\omega_0)}^{\textbf{Phase}}}$$

By linearity a sum of frequencies go out of the LTI filter only with different amplitude and phase.

Frequency Response

Discrete time

$$x[n] = \sum_{k=-\infty}^{\infty} a_k e^{ik\omega_0 n} \longrightarrow \boxed{h[n]} \longrightarrow y[n] = \sum_{k=-\infty}^{\infty} H(k\omega_0) a_k e^{ik\omega_0 n}$$

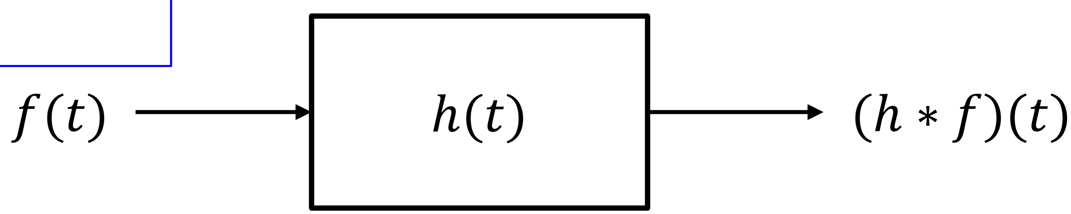
$$a_k \rightarrow \underbrace{H(k\omega_0)}_{\textbf{Gain}} a_k \qquad\qquad H(k\omega_0) = \underbrace{|H(k\omega_0)|}_{\textbf{Amplitude}} e^{\overbrace{i\measuredangle H(k\omega_0)}^{\textbf{Phase}}}$$

By linearity a sum of frequencies go out of the LTI filter
only with different amplitude and phase.

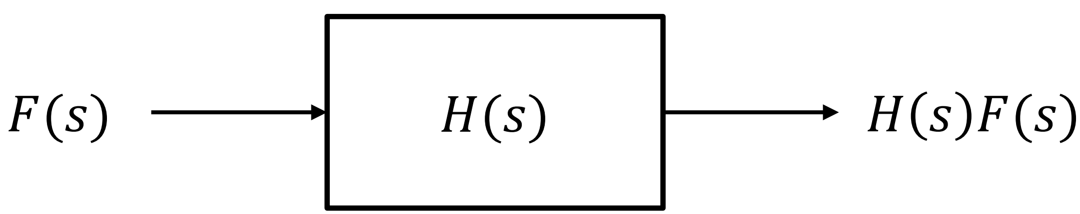
Transfer Functions

h(t) : impulse response (CT)
H(s) : transfer function (stationary and transient frequency response) of the filter/system, i.e Laplace transform of h(t)

Time domain



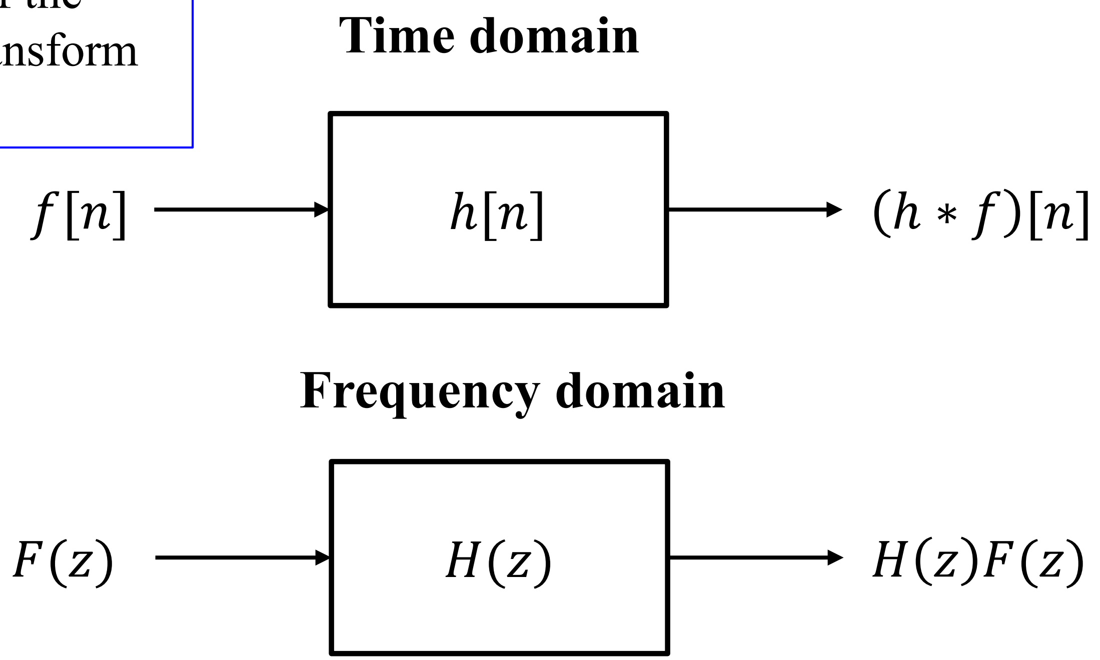
Frequency domain



Reminder: a convolution in the time domain is a multiplication in the frequency domain

Transfer Functions

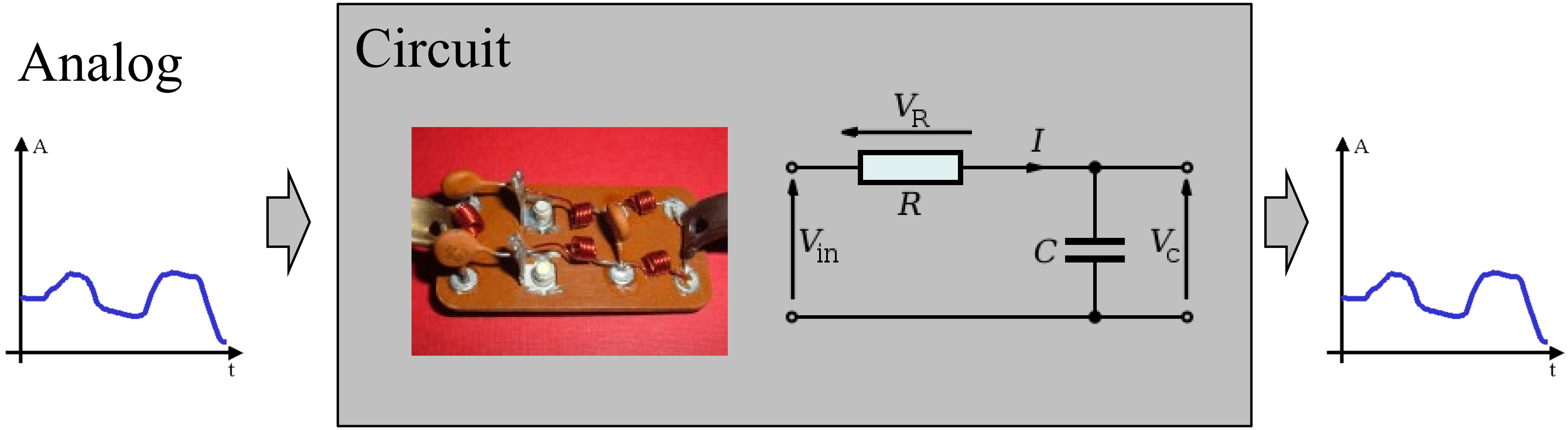
h[n] : impulse response (DT)
H(z) : transfer function (stationary and transient frequency response) of the filter/system, i.e. Z-transform of h[n]



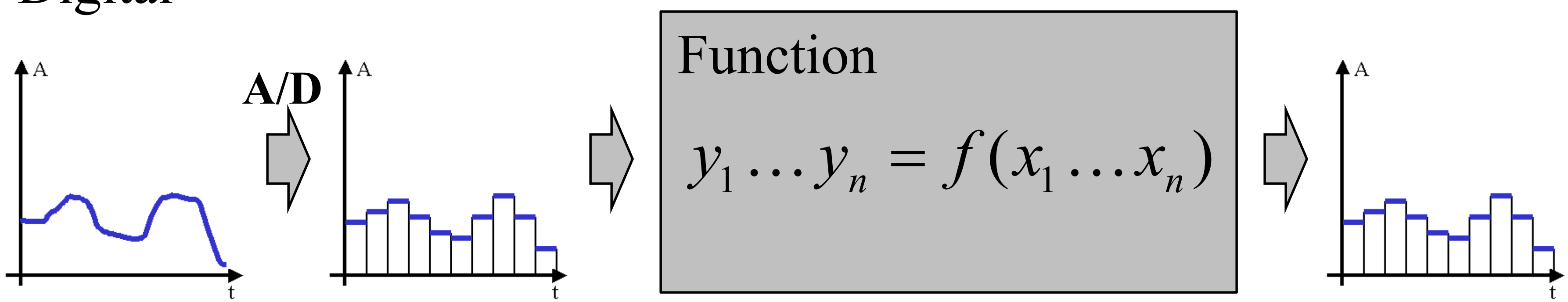
Reminder: a convolution in the time domain is a multiplication in the frequency domain

Filters as System Examples

Analog



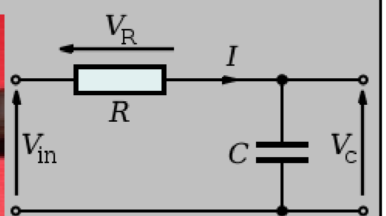
Digital



Transfer Functions of Filters

Analog

Circuit

Laplace Transf.

Numerator

$$H(s) = \frac{v_c}{v_{in}} = \frac{\overbrace{1}}{\underbrace{1+RCs}}$$

Denominator

Digital

Function

$$y_1 \ldots y_n = f(x_1 \ldots x_n)$$

Z Transf.

Numerator

$$H(z) = \frac{\overbrace{b_0 + b_1 z^{-1} + \ldots + b_N z^{-N}}}{\underbrace{1 + a_1 z^{-1} + \ldots + b_M z^{-M}}}$$

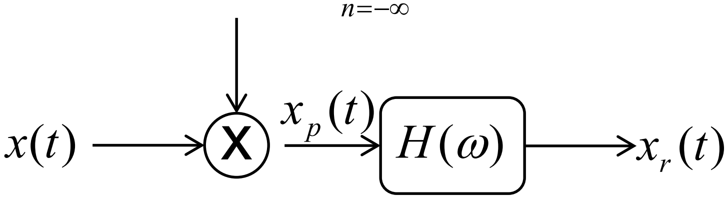
Denominator

Motivating Examples for Filters

Filters for Signal Reconstruction

From W3, s. 26

$$p(t)=\sum_{n=-\infty}^{+\infty}\delta(t-nT)$$



spectrum of original signal

spectrum of sampled signal

sampling angular frequency

$\omega_s > 2\ \omega_m$

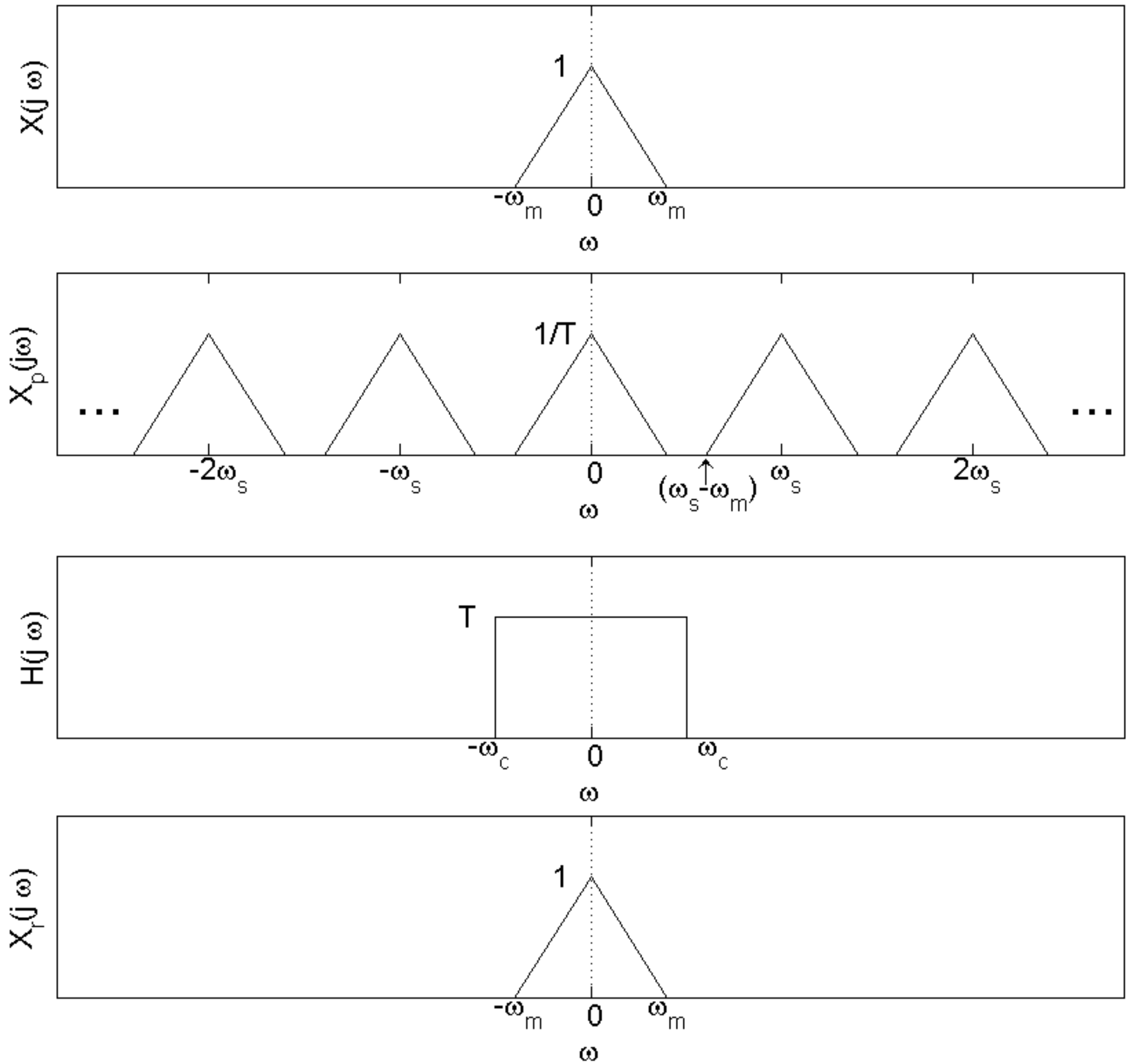
filtering

filter cut-off angular frequency

$\omega_m < \omega_c < (\omega_s - \omega_m)$

$$X_r(\omega)=X_p(\omega)H(\omega)$$

spectrum of reconstructed signal

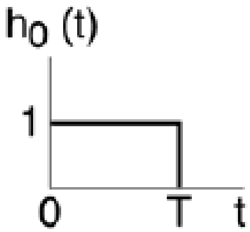


Reconstruction Summary – Time Domain

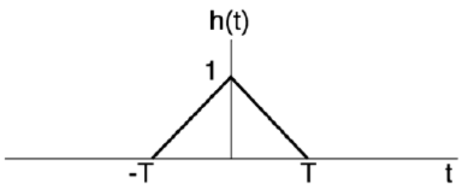
From W3, s. 32

The reconstructed signal $x_r(t)$ is obtained through a convolution between the sampled signal $x_p(t)$ with period T and one of the following three interpolation functions h(t).

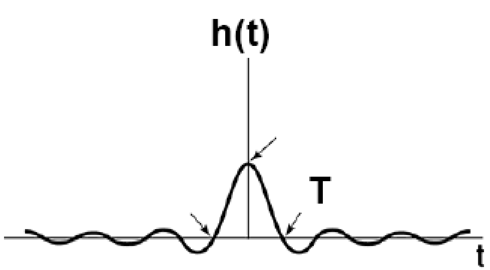
1. Zero-order hold (ZOH)



2. First-order hold (FOH)



3. Whittaker-Shannon



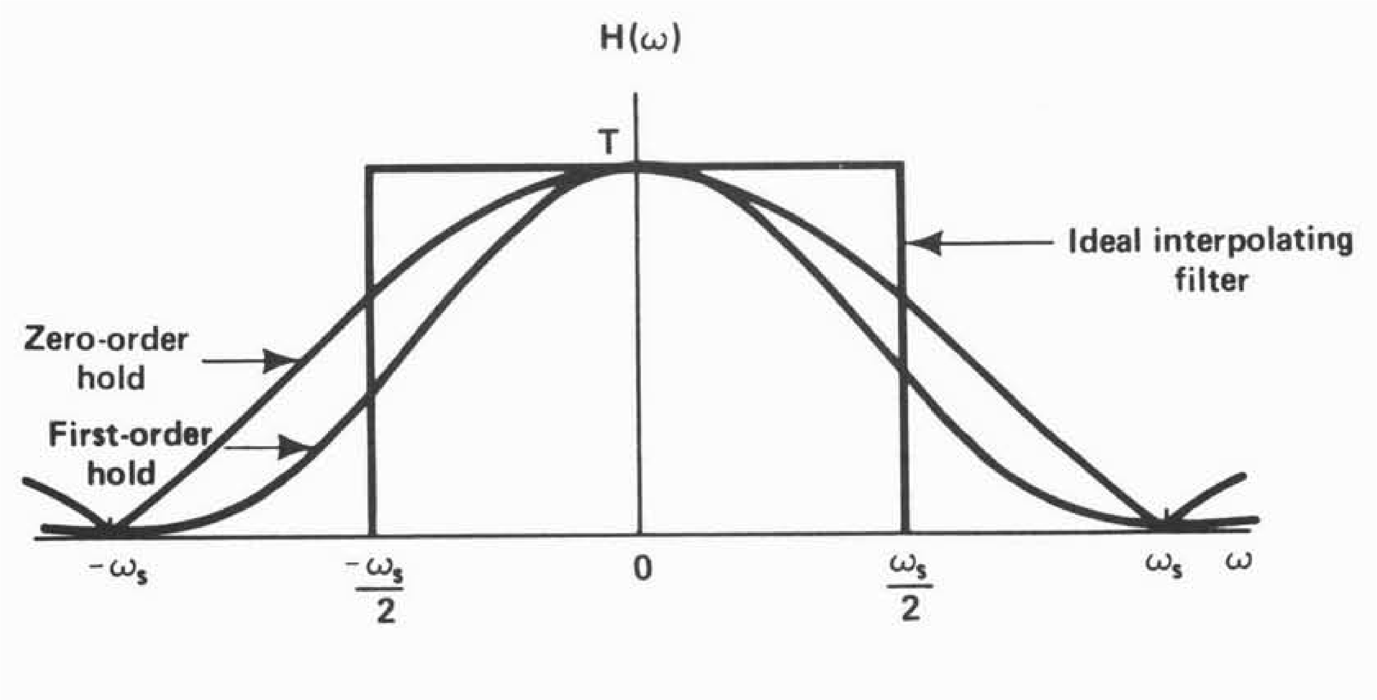
Computational cost

From Prof. A. S. Willsky, Signals and Systems course

Reconstruction Summary – Frequency Domain

From W3, s. 33

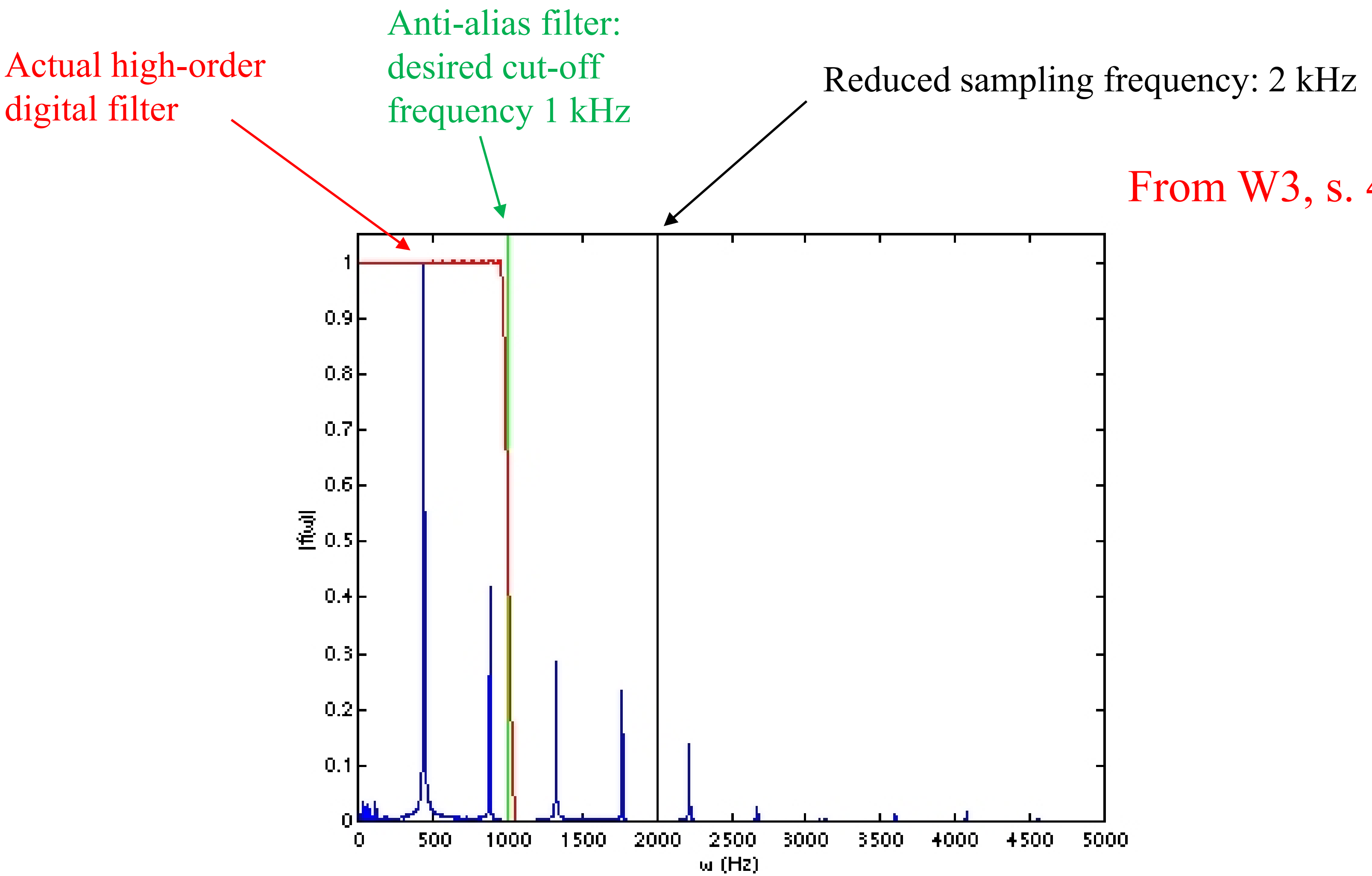
The spectrum of the reconstructed signal $X_r(\omega)$ is obtained through a multiplication between the spectrum of the sampled signal $X_p(\omega)$ with angular sampling frequency ω_s and one of the following three low-pass filters.



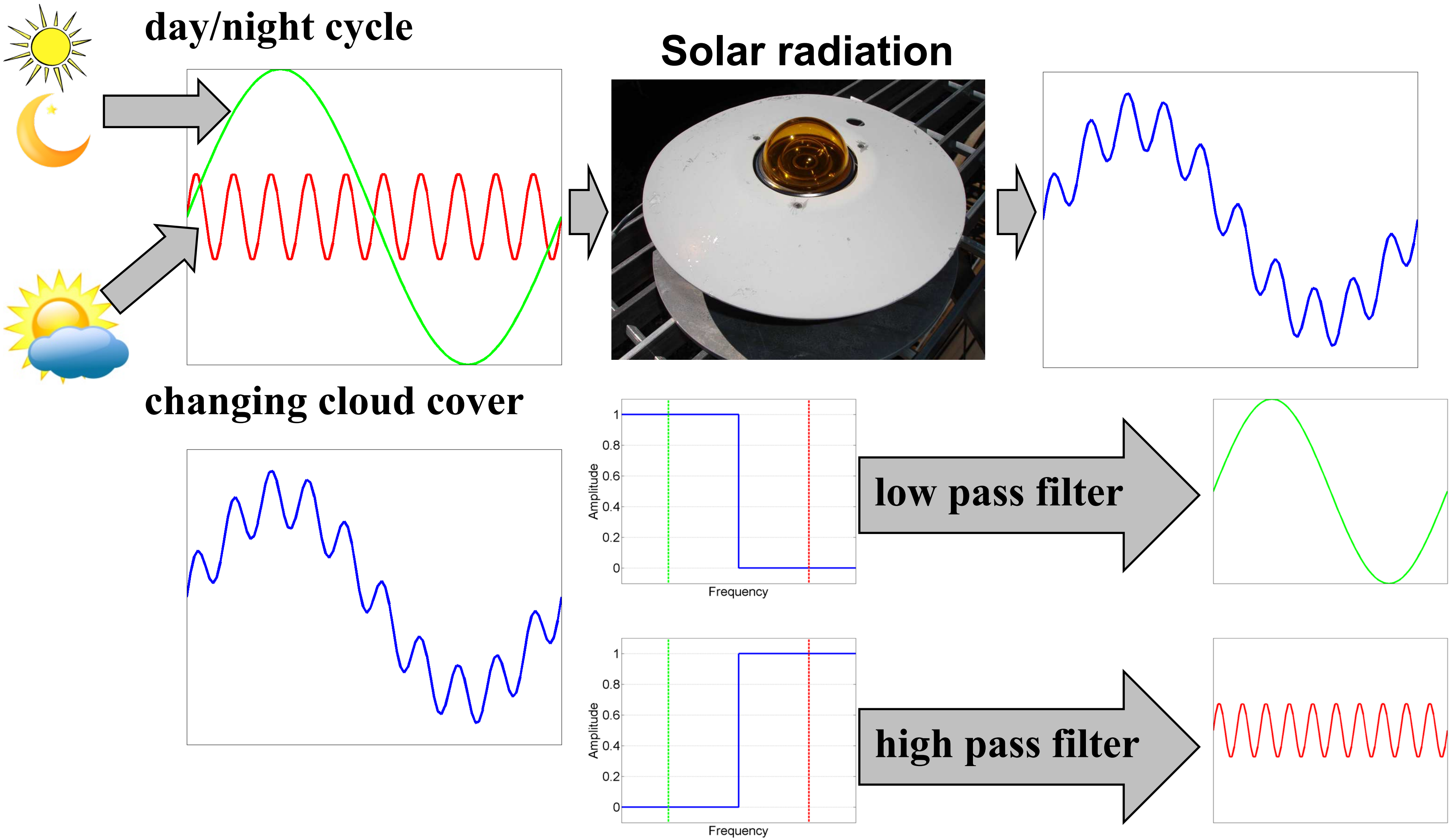
From Prof. A. Oppenheim, Signals and Systems course

Anti-Aliasing Filters

From W3, s. 42



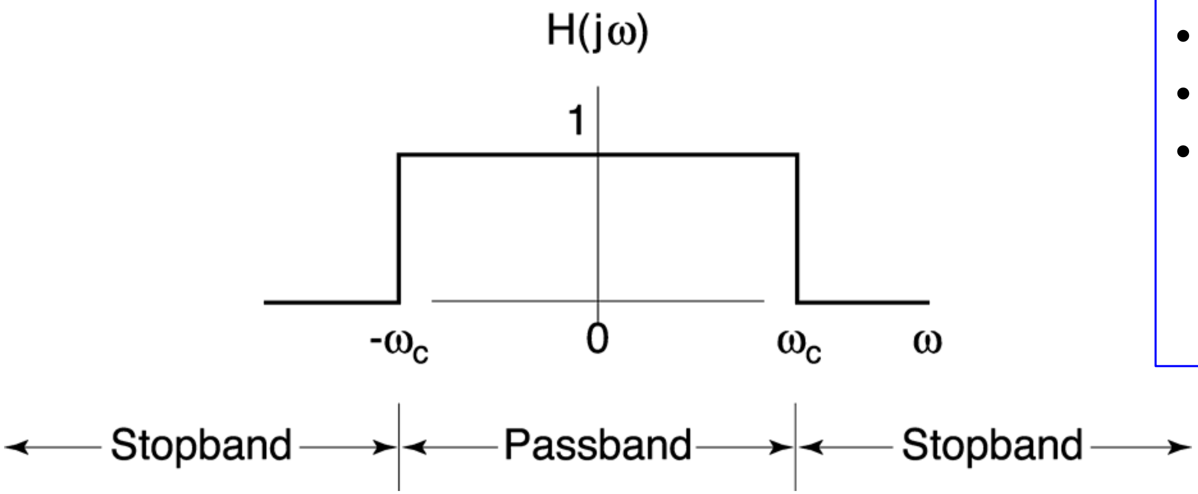
Filtering Noisy Signals



Basic Filters

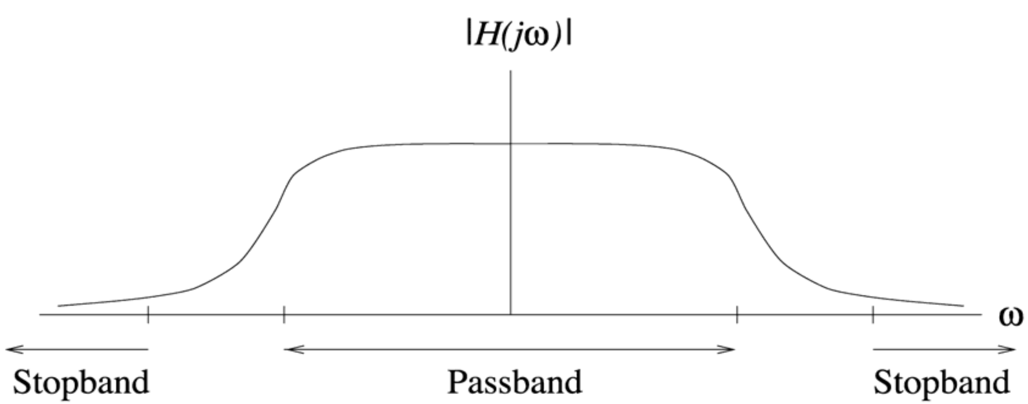
Low-Pass Filter

Idealized response in the (angular) frequency domain:



Notes:
- $H(j\omega) = H(i\omega) = H(\omega)$
- ω_c: cut-off frequency
- $|H| = 1$ and $\angle H = 0$ for ideal filters in the passband, no need for the phase plot.

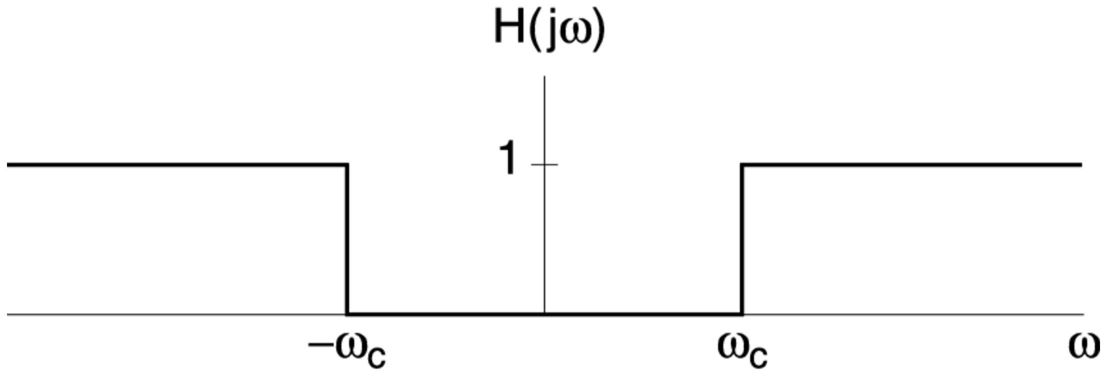
Realistic response in the (angular) frequency domain:



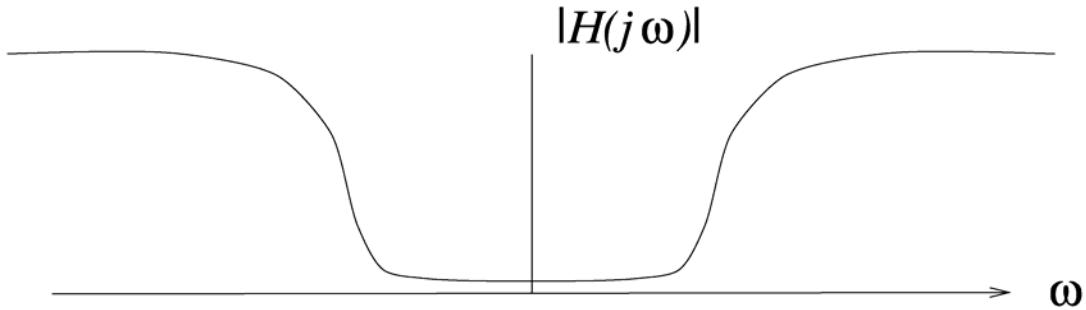
Notes:
$|H(j\omega)|$: amplitude

High-Pass Filter

Idealized response in the (angular) frequency domain:

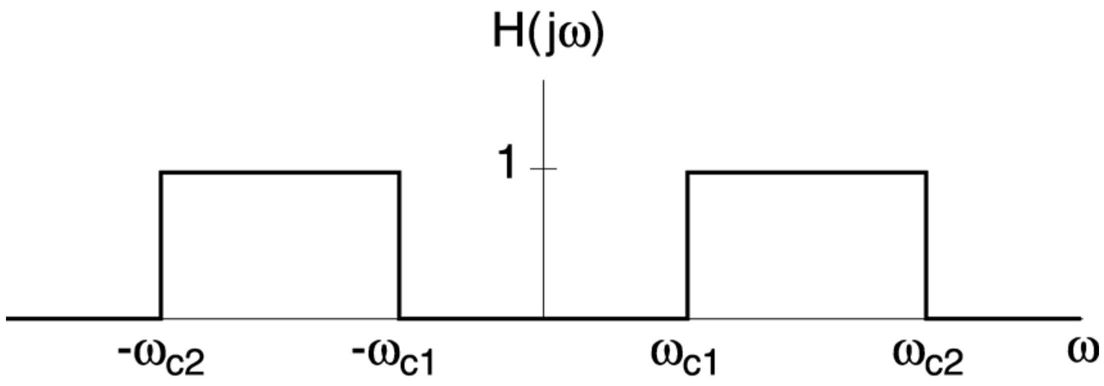


Realistic response in the (angular) frequency domain:



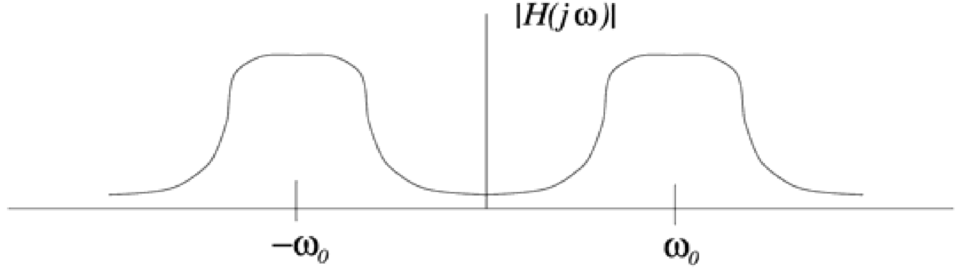
Band-Pass Filter

Idealized response in the (angular) frequency domain:



ω_{c1}: lower cut-off frequency
ω_{c2}: upper cut-off frequency

Realistic response in the (angular) frequency domain:



Notes:
a Band-Stop Filter reverses passing and stopping bands w.r.t. a band-pass filter

Bode Plots

From Transfer Functions to Bode Plots

Assume transfer function: $H(s) = \frac{Num(s)}{Den(s)} = \mathrm{A}\prod \frac{(s-x_n)^{a_n}}{(s-y_n)^{b_n}}$

where x_n, y_n constants, a_n, $b_n > 0$

Frequency response: $\mathrm{s} = i\omega$

Bode **magnitude**: $|H(s = i\omega)| = |H(i\omega)| = |H(\omega)|$

Bode **phase**: $\measuredangle H(s = i\omega) = \measuredangle H(i\omega) = \measuredangle H(\omega)$

Zero (numerator = 0): every value of s where $\omega = |x_n|$

Pole (denominator = 0): every value of s where $\omega = |y_n|$

Decibel

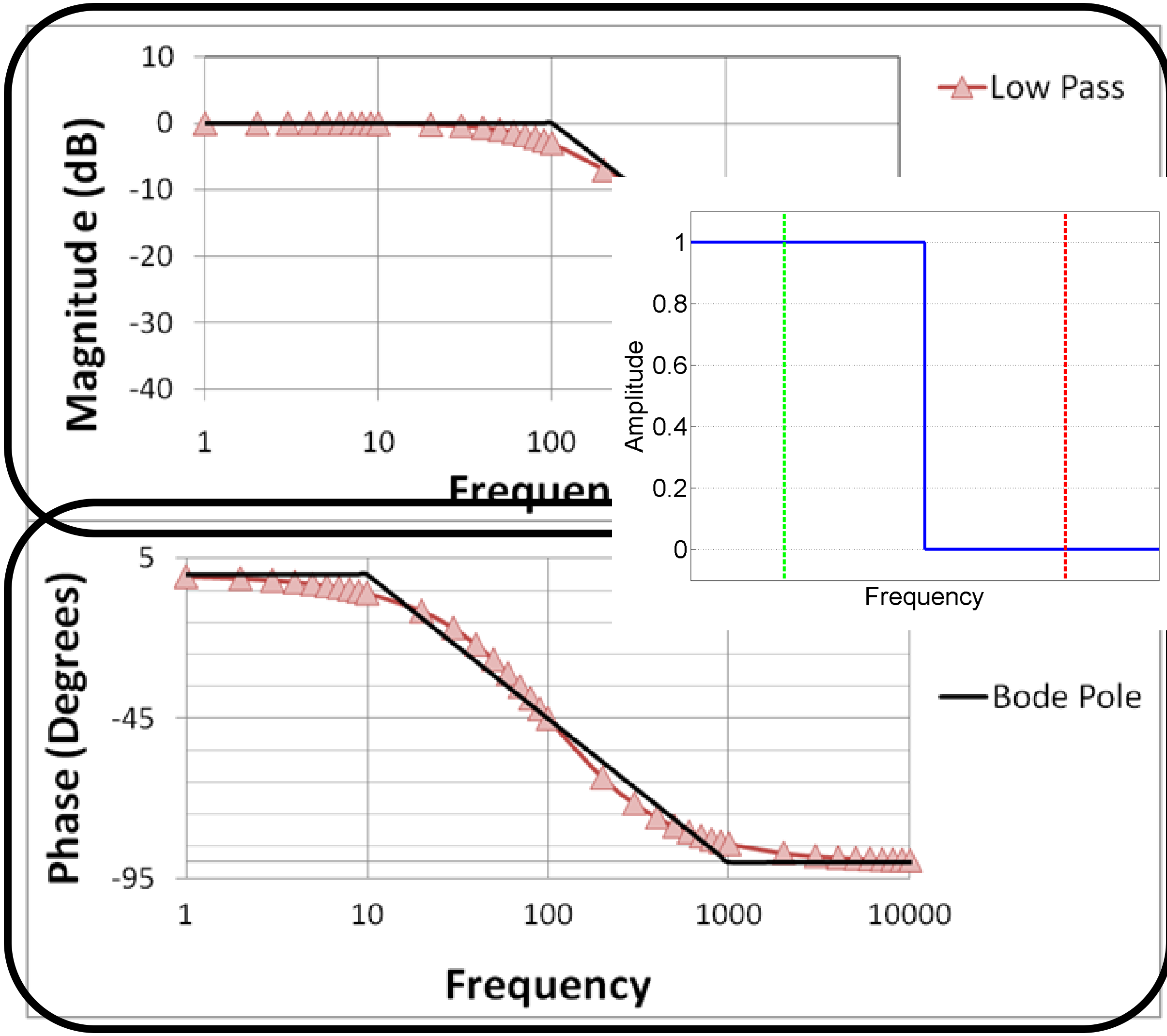
$$G_{dB} = 20\log_{10}\left(\frac{V_2}{V_1}\right)$$

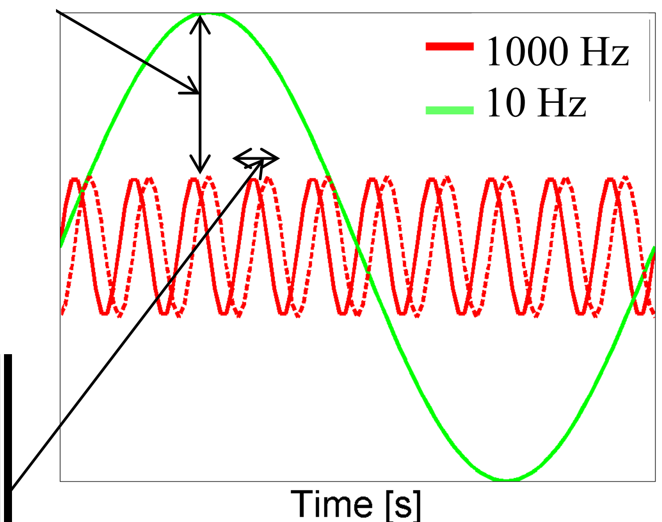
$$V_2 > V_1 \rightarrow G_{dB} > 1(gain)$$

$$V_2 < V_1 \rightarrow G_{dB} < 1(damping)$$

Source of sound	Sound pressure	Sound pressure level
	pascal	dB re 20 µPa
Jet engine at 30 m	630 Pa	150 dB
Rifle being fired at 1 m	200 Pa	140 dB
Threshold of pain	100 Pa	130 dB
Hearing damage (due to short-term exposure)	20 Pa	approx. 120 dB
Jet at 100 m	6 – 200 Pa	110 – 140 dB
Jack hammer at 1 m	2 Pa	approx. 100 dB
Hearing damage (due to long-term exposure)	6×10^{-1} Pa	approx. 85 dB
Major road at 10 m	2×10^{-1} – 6×10^{-1} Pa	80 – 90 dB
Passenger car at 10 m	2×10^{-2} – 2×10^{-1} Pa	60 – 80 dB
TV (set at home level) at 1 m	2×10^{-2} Pa	approx. 60 dB
Normal talking at 1 m	2×10^{-3} – 2×10^{-2} Pa	40 – 60 dB
Very calm room	2×10^{-4} – 6×10^{-4} Pa	20 – 30 dB
Leaves rustling, calm breathing	6×10^{-5} Pa	10 dB
Auditory threshold at 1 kHz	2×10^{-5} Pa	0 dB

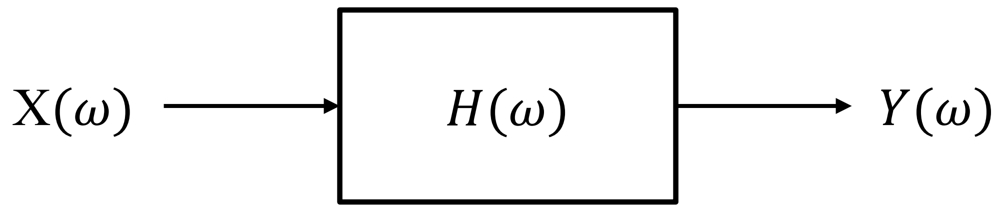
Bode Plot: The Example of a Low-Pass Filter





not to scale !

Bode Plots – Why handy?



Frequency domain

$$Y(\omega) = H(\omega)X(\omega)$$

$$|Y(\omega)| = |H(\omega)||X(\omega)|$$ Amplitude

$$\measuredangle Y(\omega) = \measuredangle H(\omega) + \measuredangle X(\omega)$$ Phase

$$\log|Y(\omega)| = \log|H(\omega)| + \log|X(\omega)|$$

Note: this is valid also for cascaded blocks of LTI systems (e.g., cascaded filters)

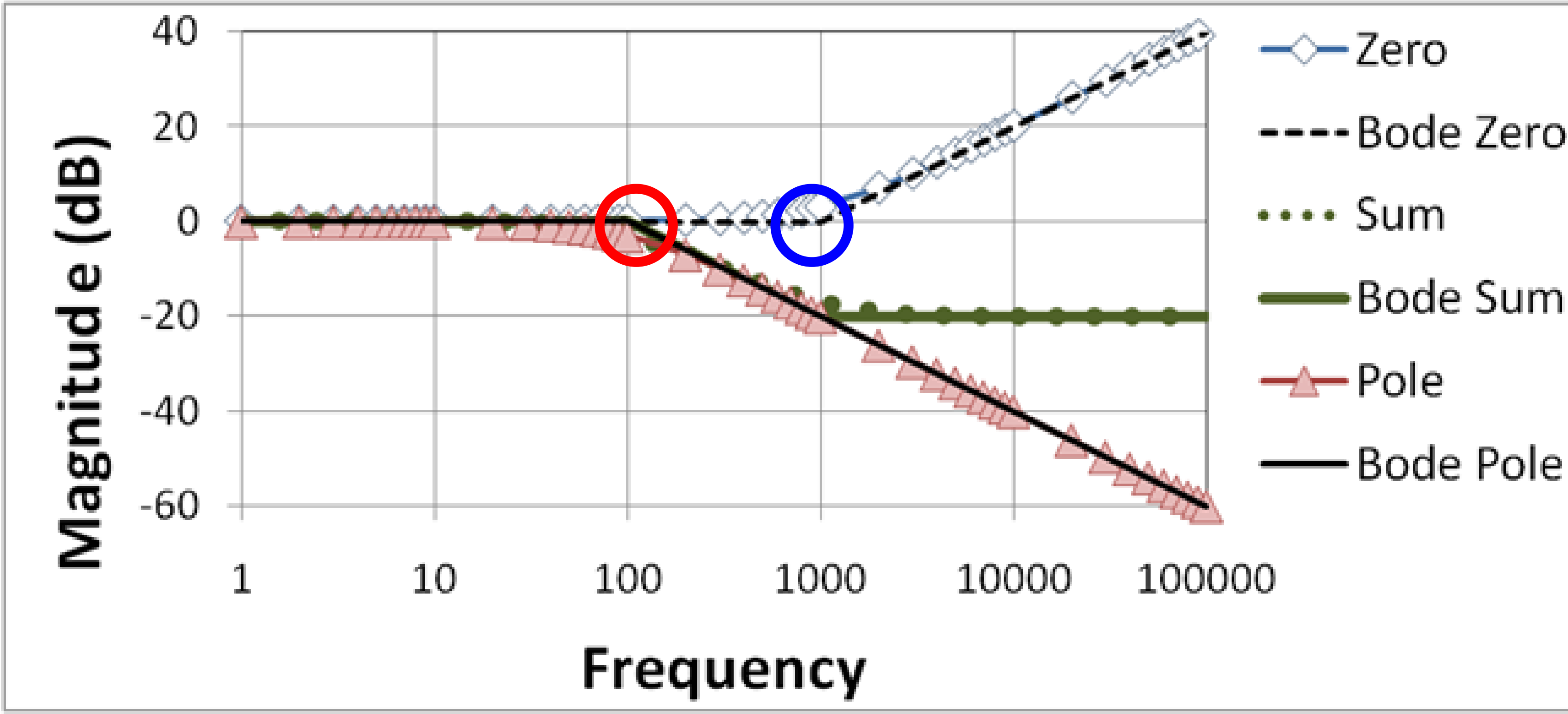
Bode Plot – Why only positive and log scale for ω as well?

- If impulse response h(t) real, then $|H(\omega)|$ **even** function of ω and $\measuredangle H(\omega)$ **odd** function of ω
- Therefore plots for negative ω can be straightforwardly obtained from those of positive ω, so disregarded
- Log scale for frequency allows for covering a wider range of possible input frequencies on the same plot

Bode Plot - Rules

- Zero (numerator = 0)
  - Amplitude: +20 dB/decade
  - Phase: +90º; +45º/decade, starting 1 decade before zero
- Pole (denominator = 0)
  - Amplitude: -20 dB/decade
  - Phase: -90º; -45º/decade, starting 1 decade before pole

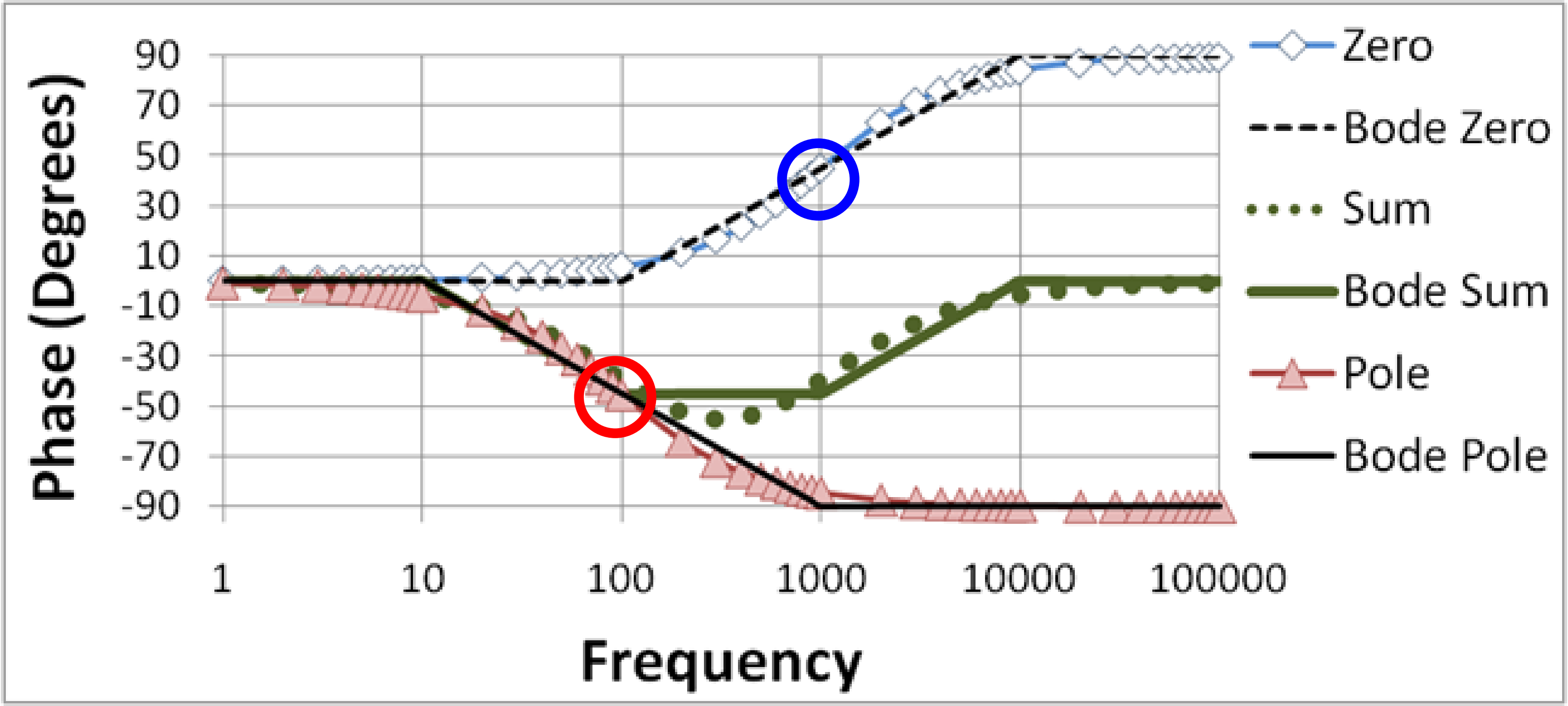
Bode Plot (Magnitude)



Zero (numerator = 0) Amplitude: +20 dB/decade

Pole (denominator = 0) Amplitude: -20 dB/decade

Bode Plot (Phase)



Zero (numerator = 0): +90º; 45º/decade, starting 1 decade before zero

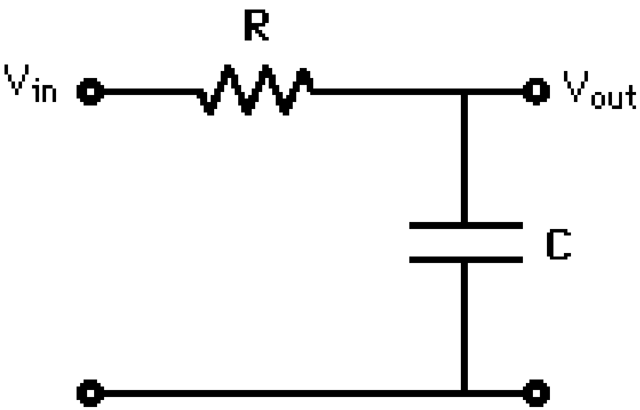
Pole (denominator = 0): -90º; -45º/decade, starting 1 decade before pole

Examples of First Order Analog Filters

Low Pass Filter - RC circuit

1 pole s = -1/RC

i.e. $\omega = \left|\frac{i}{RC}\right| = \frac{1}{RC}$



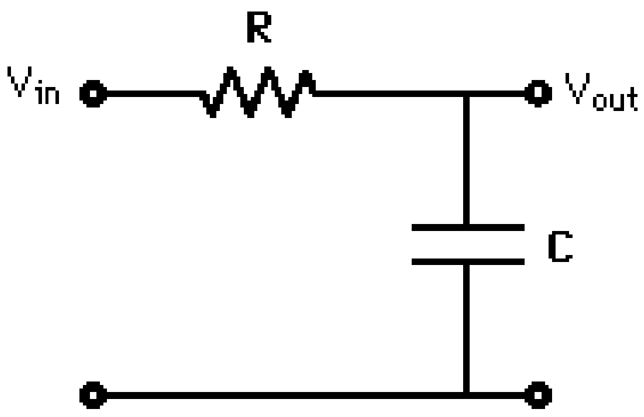
$$H(s) = \frac{1}{1+sRC}$$

$$H(s=i\omega) = H(\omega) = \frac{1}{1+i\omega RC}$$

Bode **magnitude**: $$|H(\omega)| = \left|\frac{1}{1+iRC\omega}\right| = \frac{1}{|1+i\omega RC|} = \frac{1}{\sqrt{1+\omega^2R^2C^2}}$$

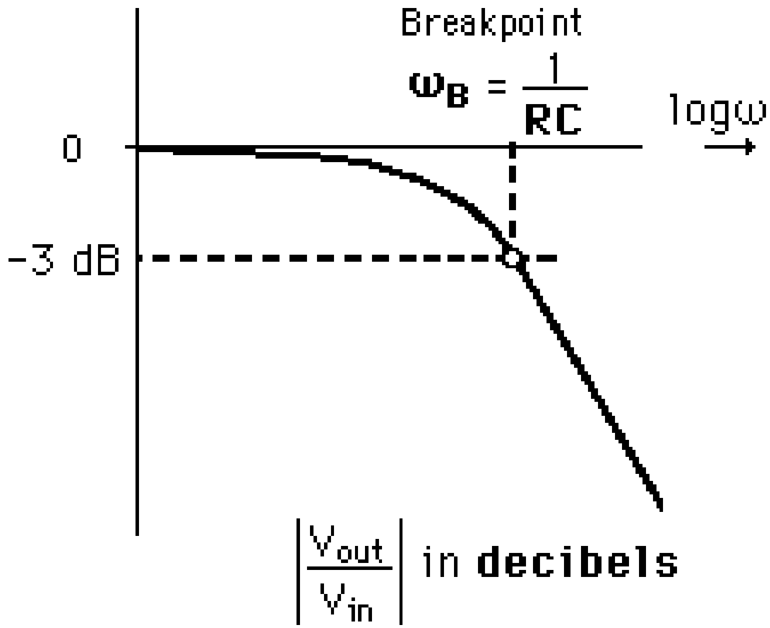
Bode **phase**: $$\measuredangle H(\omega) = -\tan^{-1}\left(\frac{Im[H(\omega)]}{Re[H(\omega)]}\right) = -\tan^{-1}(\omega RC)$$

Low Pass Filter - RC circuit



$$|V_{out}| = |V_{in}| \frac{1}{\sqrt{1 + \omega^2 R^2 C^2}}$$

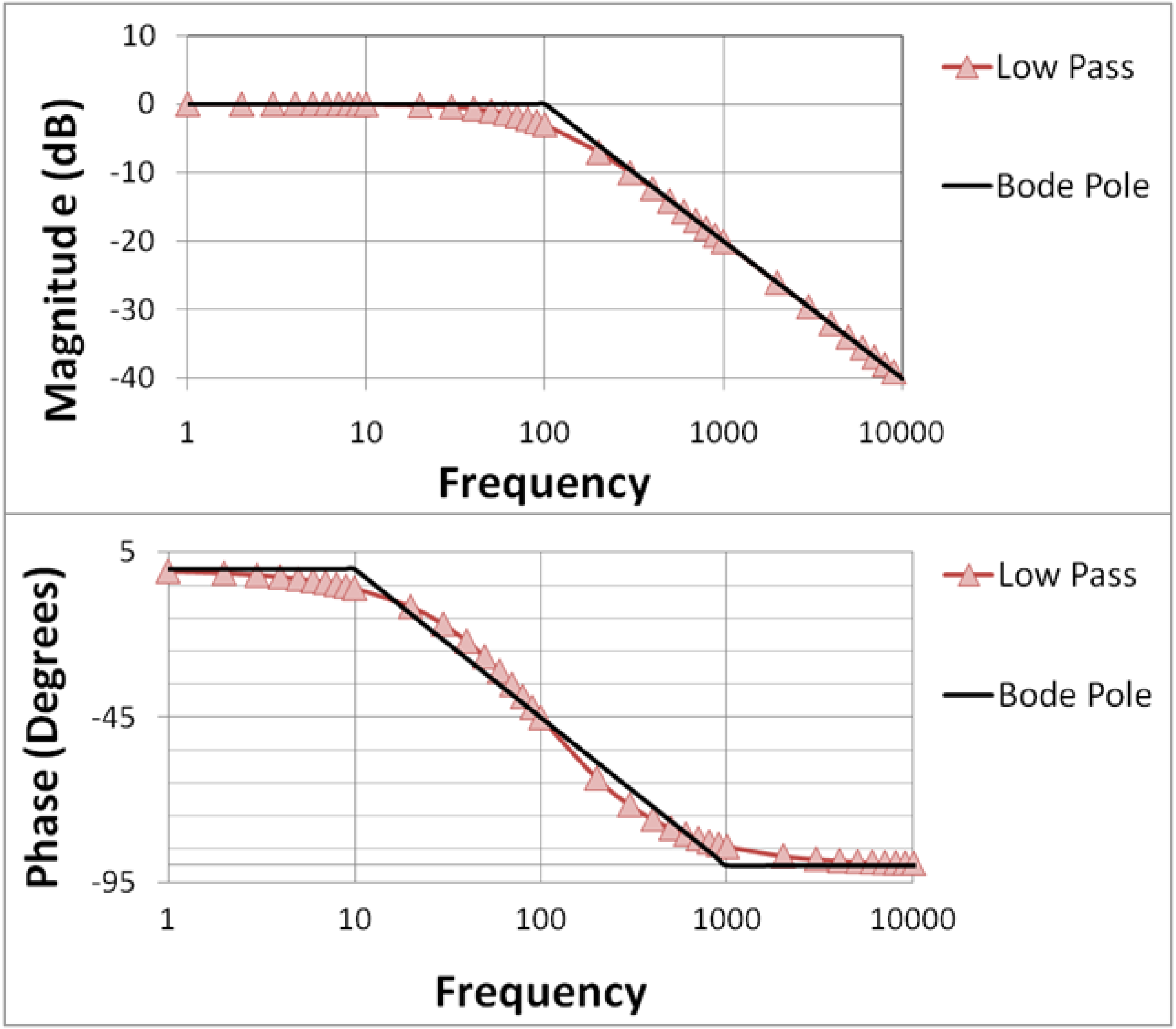
Break or cut-off frequency



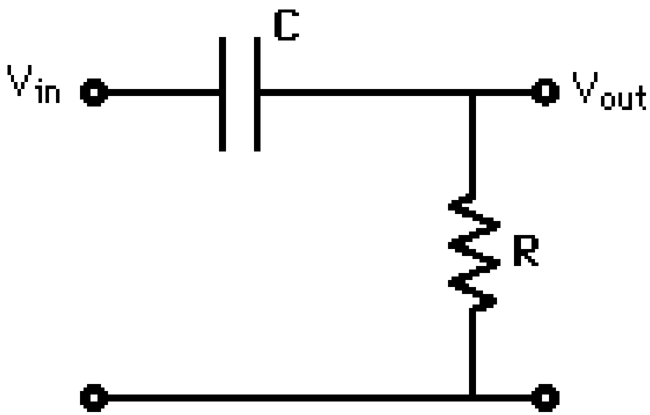
1 pole at $\omega = 1/RC$

Note: -3dB = about 71% of undamped amplitude

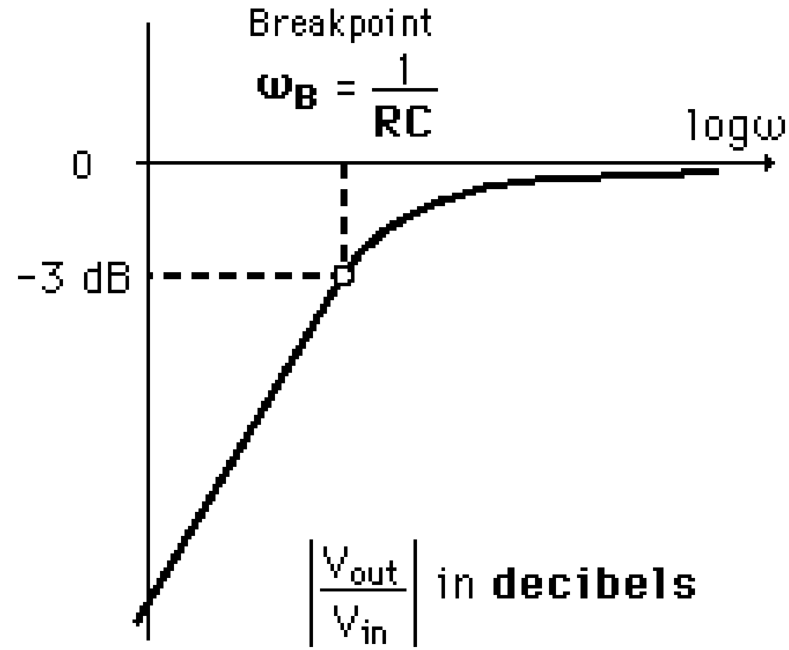
Bode Plot – First Order Low Pass Filter



High-Pass Filter – RC circuit

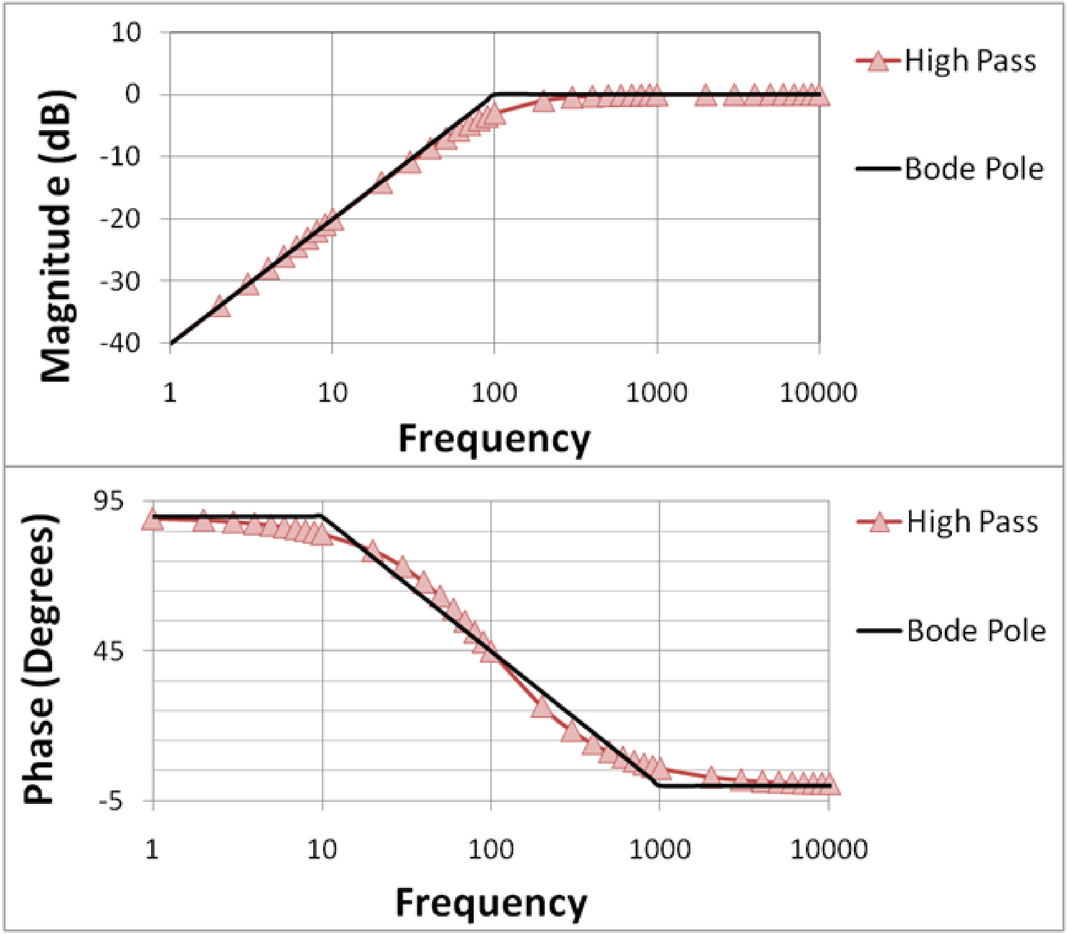


$$|V_{out}| = |V_{in}| \frac{\omega RC}{\sqrt{1 + \omega^2 R^2 C^2}}$$



1 zero at $\omega = 0$
1 pole at $\omega = 1/RC$

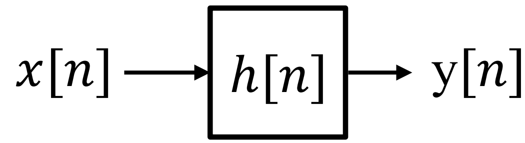
Bode plot –
First Order High-Pass Filter



Digital Filters

The General Case

Difference equation:



$$y[n] = -\sum_{k=1}^{N} a_k y[n-k] + \sum_{k=0}^{M} b_k x[n-k]$$

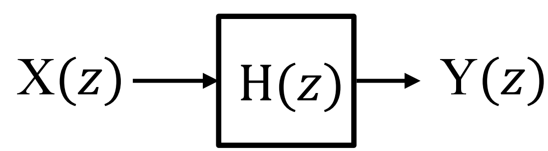
Z-transform:

$$Y(z) + \sum_{k=1}^{N} a_k Y(z) z^{-k} = \sum_{k=0}^{M} b_k X(z) z^{-k}$$

s. 16, Z-transform definition

Transfer function:

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 + \sum_{k=1}^{N} a_k z^{-k}}$$



An Example

Difference equation:

$$y[n] = x[n] + 2x[n-1] + x[n-2] - \frac{1}{4}y[n-1] + \frac{3}{8}y[n-2]$$

Z-transform:

$$Y(z) + \frac{1}{4}Y(z)z^{-1} - \frac{3}{8}Y(z)z^{-2} = X(z) + 2X(z)z^{-1} + X(z)z^{-2}$$

Transfer function:

$$H(z) = \frac{\mathrm{Y}(z)}{X(z)} = \frac{1 + 2z^{-1} + z^{-2}}{1 + \frac{1}{4}z^{-1} - \frac{3}{8}z^{-2}} = \frac{z^2 + 2z + 1}{z^2 + \frac{1}{4}z - \frac{3}{8}} = \frac{(z+1)^2}{\left(z - \frac{1}{2}\right)\left(z + \frac{3}{4}\right)}$$

Note: poles and zeros of H(z) can be represented in the zero-pole plot (complex Z-plane) for further analysis (e.g., stability).

Frequency Response

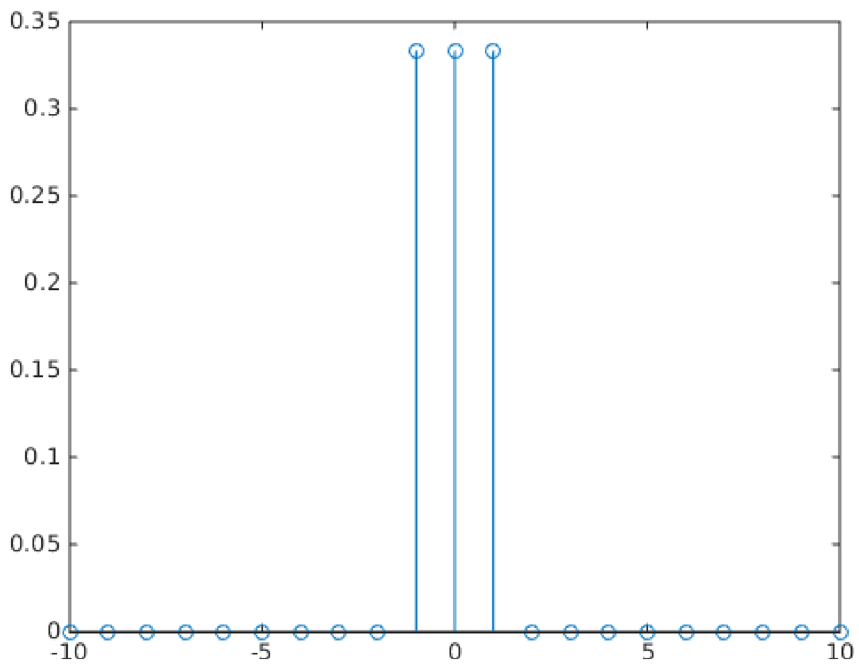
- Bode plots have been defined for continuous-time systems (e.g., analog filters)
- With z $= e^{i\omega}$ (see s. 16), the Z-transform degrades to a DTFT (as the Laplace transform was degrading to the FT with s $= i\omega$)
- We can therefore calculate the frequency response of the corresponding discrete-time system (e.g., a digital filter) with the transfer function $H(e^{i\omega})$
- A FFT is then typically used to calculate numerically the frequency response on computers
- Matlab has a dedicated function for this: freqz

Nonrecursive Discrete-Time Filters

- $a_k = 0$ for all k
- $y[n] = \sum_{k=-N}^{M} b_k \, x[n-k]$
- Finite Impulse Response (FIR)
- Unrelated to continuous time filtering
- If $b_k \neq 0$ for any n<0 then noncausal

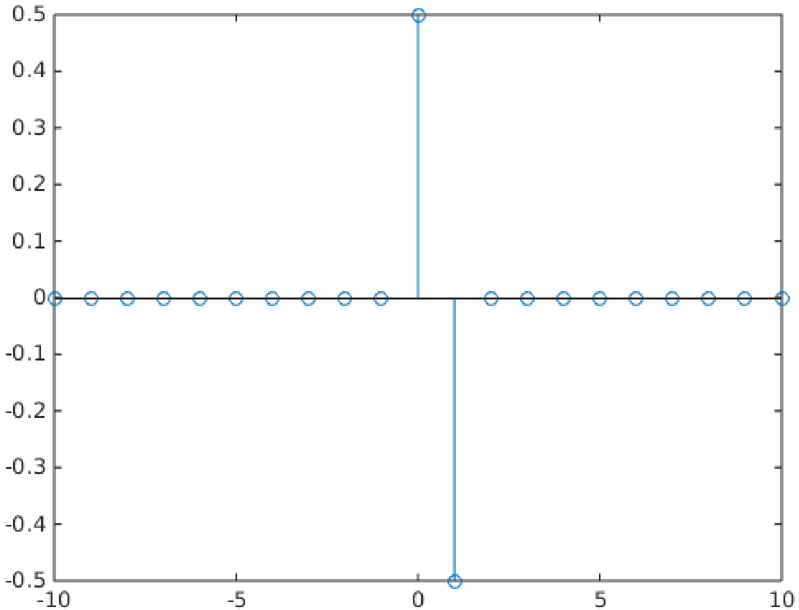
Example of FIR Filter: 3-Point Moving Averaging

- $y[n] = \frac{1}{3}(x[n-1] + x[n] + x[n+1])$
- $h[n] = \frac{1}{3}(\delta[n-1] + \delta[n] + \delta[n+1])$ *(FIR)*



Example of FIR Filter: Simple High Pass Filter

- $\mathrm{y}[\mathrm{n}] = \frac{1}{2}(x[n] - x[n-1])$
- $\mathrm{h}[\mathrm{n}] = \frac{1}{2}(\delta[n] - \delta\,[n-1])$ *(FIR)*



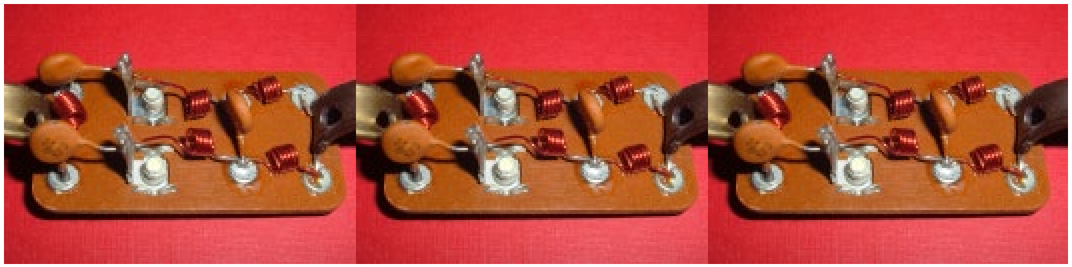
Order and Types of Filters

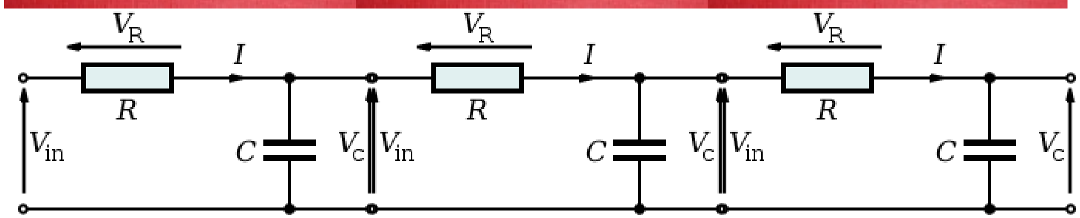
Filter Order and Type

- Several filters exists and are defined by the polynomials at the numerator/denominator (Finite Impulse Response, Bessel, Butterworth, Tschebishev, etc.)
- 1st order is equivalent to 20dB per decade
- Each successive order adds 20dB per decade
- Filter with a high order are closer to the ideal filter (rectangular function)

Filter Order

Analog





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Filter order: 3
++ faster cutoff
-- more components
-- higher power consumption

Digital

$$y[n] = b_0 x[n]$$ Filter order: 0

$$y[n] = b_0 x[n] + b_1 x[n-1]$$ Filter order: 1

$$y[n] = b_0 x[n] + b_1 x[n-1] + b_2 x[n-2]$$ Filter order: 2

…

++ faster cutoff
-- more computation
-- higher power consumption

EPFL

Disal

Conclusion

Take-Home Messages

- Filters allow a number of operations (e.g., noise removal, contrast enhancement, anti-aliasing, etc.)
- Their response can be represented in time (impulse response) and frequency domain (frequency response, transfer function)
- They are often easier to design and analyze in the frequency domain
- Bode plots allows for analysis of analog filters' frequency response; digital filters' frequency response is obtained first by degrading the transfer function to the discrete-time and applying an equivalent plotting function
- Filters are characterized by different order and coefficient distributions
- Programmable digital components (e.g., microcontrollers, DSPs) allow for easy encoding of digital filters

Additional Reading

Books

- J. H. McClellan, R. W. Schafer, M. A. Yoder “DSP First: A Multimedia Approach”, Prentice Hall, 1999.
- A. Oppenheim and A. S. Willsky with S. Nawab, “Signals and Systems”, Prentice Hall, 1997.