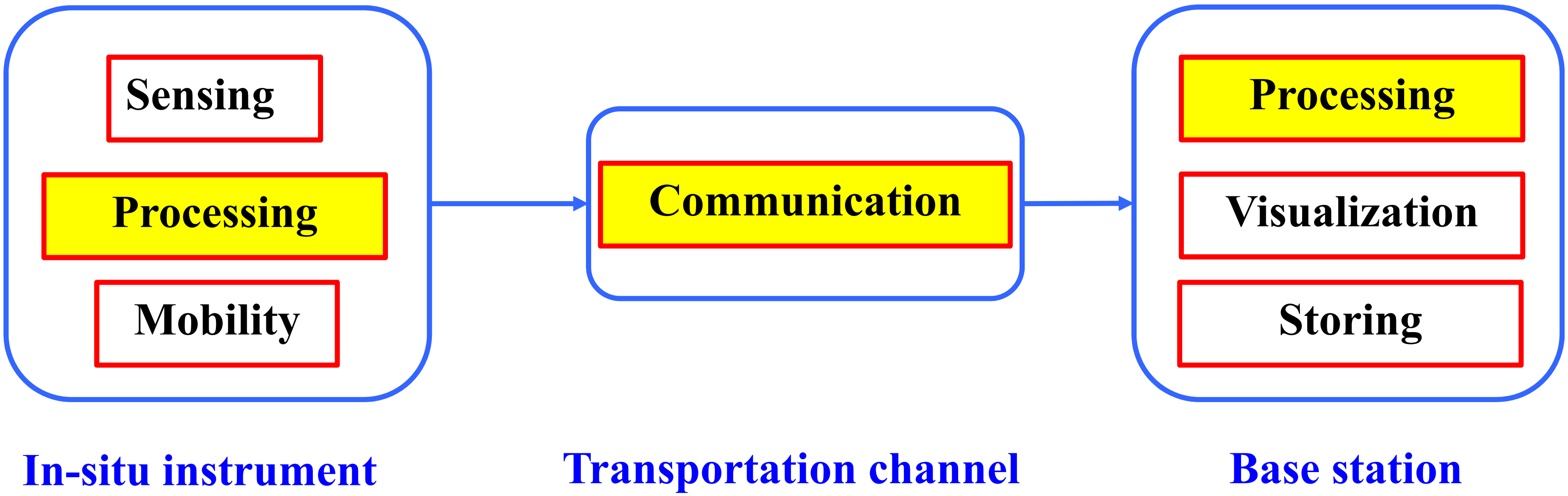
Signals, Instruments, and Systems – W3

Introduction to Signal Processing – Sampling, Reconstruction, and Aliasing

Motivation from Week 1 Lecture

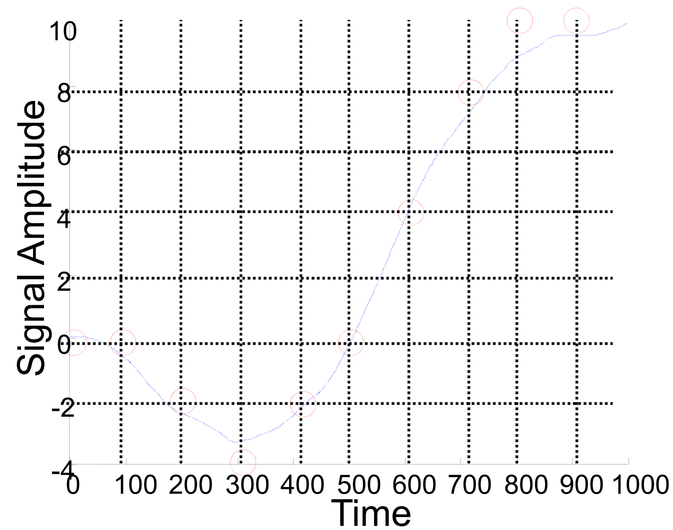


Highlighted blocks are those mainly leveraging the content of this lecture.

Sampling

Analog-Digital Converter (ADC)

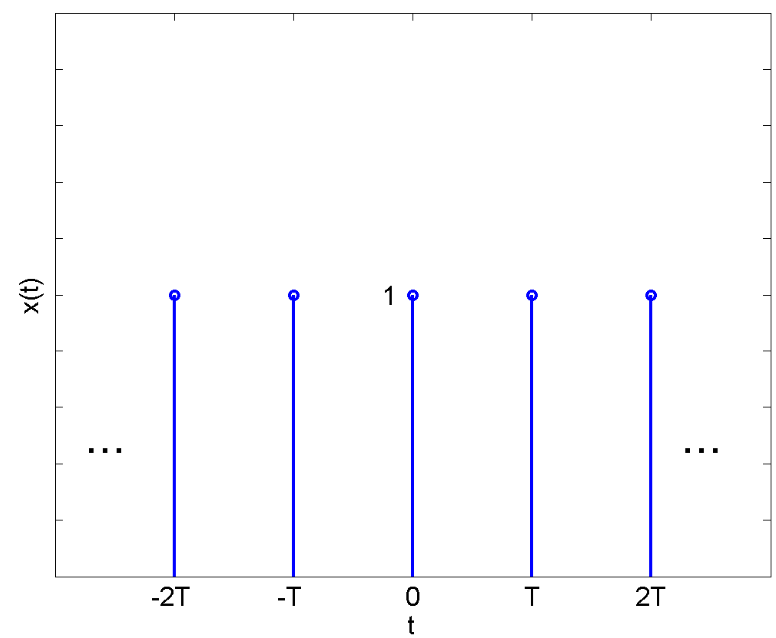
- Transforms continuous analog signal into series of values
- Two key elements
  - **Sampling** (in time)
  - **Quantization** (of values)



y[n]=0 0 -2 -4 -2 0 4 8 10 10

Periodic Train of Dirac Impulses

$$x(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

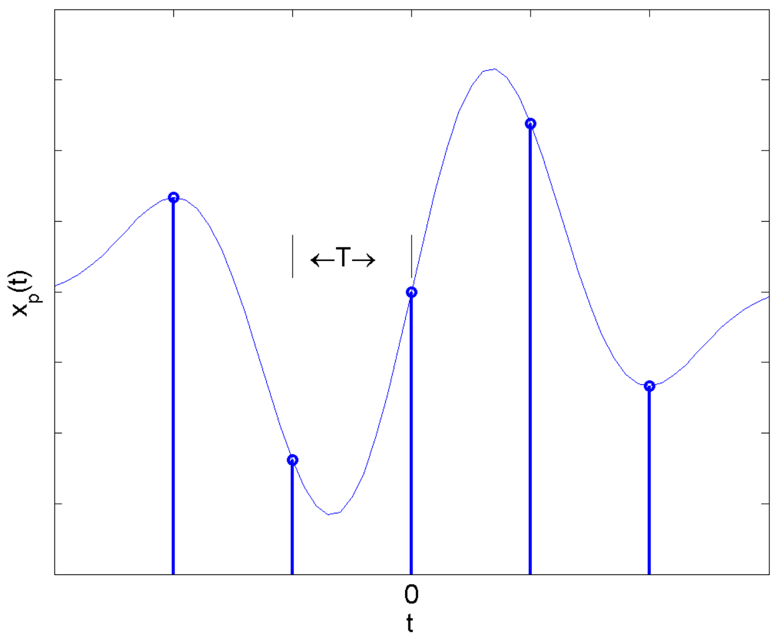


Sampling in Time Domain

$x(t)$

$p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$ Periodic train of Dirac impulses

$$x_p(t) = x(t)p(t) = \sum_{n=-\infty}^{\infty} x(t)\delta(t - nT) = \sum_{n=-\infty}^{\infty} x(nT)\delta(t - nT)$$



T = sampling period
f_s = 1/T = sampling frequency

Sampling in the Frequency Domain

- It would be nice to understand what does it mean sampling in the frequency domain so that we can leverage this representation for further reasoning (e.g., filter design)
- Multiplication in time domain means convolution in frequency domain but …
- How does look like a periodic train of Dirac impulses in the frequency domain?

Complex Coefficients for Arbitrary Period

From W2 (s.18)

Fourier series $$f(t) = \sum_{n=-\infty}^{\infty} C_n e^{in\omega_0 t}$$

Fourier coefficients $$C_n = \frac{1}{T}\int_{-T/2}^{T/2} f(t) e^{-in\omega_0 t} dt$$

for T periodic function

Rem: $C_n = |C_n| e^{i\varphi}$ Magnitude: $|C_n|$ Phase: φ

Fourier Transform

Non-unitary, angular frequency notation

From W2 (s.23)

$$F(\omega) = \int_{-\infty}^{+\infty} f(t) e^{-i\omega t} dt$$

Fourier Transform

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} F(\omega) e^{i\omega t} d\omega$$

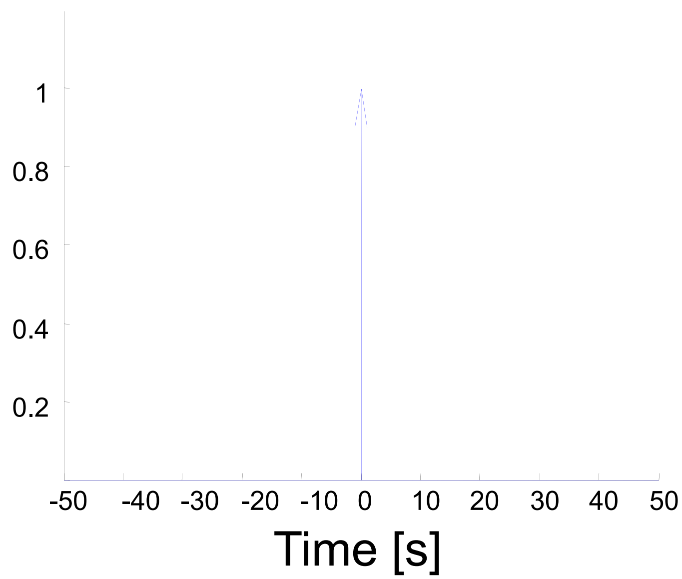
Inverse Fourier Transform

Notes:

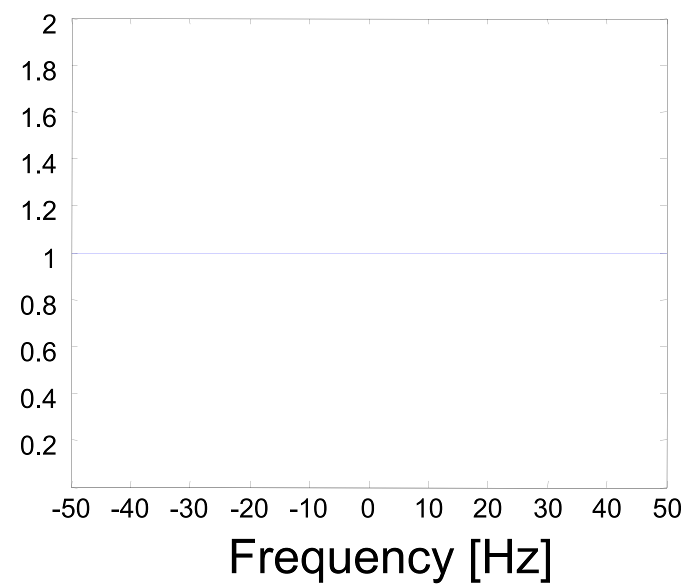
- $\omega = 2\pi\xi \rightarrow$ obtained from unitary, ordinary frequency transform with $\xi = \omega/2\pi$
- F can be replaced with f^
- In electrical engineering *i* is substituted by *j* ("i" booked for current)
- Often, in order to emphasize the frequency response aspect, the imaginary aspect of the transform is emphasized: F(ω), F(iω), or F(jω) are all equivalent notations

FT of a Dirac Delta Function (or Dirac Impulse)

From W2 (s.25)



$$f(t) = \delta(t)$$



$$\hat{f}(\xi) = 1$$

FT for Periodic Signals

- Although it generalizes to aperiodic signals, the FT can be also applied to periodic signals
- We can derive the FT of a periodic signal from its Fourier series (which have been developed for periodic signals, see W2)
- The FT of a periodic Dirac train of impulses consists of a periodic train of impulses in the frequency domain as well, with the area of the impulses proportional to the Fourier series coefficients

From [Oppenheim et al., 1997]

FT for Periodic Signals

(assume: period T, $\omega_0 = 2\pi/T$)

$$X(\omega) = 2\pi\delta(\omega - \omega_0)$$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} 2\pi\delta(\omega - \omega_0)\, e^{i\omega t} d\omega$$

W2, s. 23, IFT

$$x(t) = e^{i\omega_0 t}$$

W2, s. 30, modulation property

Linear combination of impulses equally spaced in frequency:

$$X(\omega) = \sum_{n=-\infty}^{+\infty} 2\pi C_n \delta(\omega - n\omega_0)$$

W2, s. 30, linearity property

$$x(t) = \sum_{n=-\infty}^{+\infty} C_n e^{in\omega_0 t}$$

Compare with W2, s. 18, Fourier series

FT for Periodic Impulse Train

(assume: period T, $\omega_0 = 2\pi/T$)

Now consider the periodic impulse train of before:

$$x(t) = \sum_{n=-\infty}^{+\infty} \delta(t - nT)$$

Calculate the coefficient of its Fourier series:

$$C_n = \frac{1}{T}\int_{-T/2}^{+T/2} \delta(t) e^{-in\omega_0 t} dt = \frac{1}{T}$$

W2, s. 18, Fourier series coefficients

This means that each Fourier coefficient of the periodic impulse train has the same value; insert C_n in previous expression (s. 12):

$$X(\omega) = \sum_{n=-\infty}^{+\infty} 2\pi C_n \delta(\omega - n\omega_0)$$

$$X(\omega) = \sum_{n=-\infty}^{+\infty} \frac{2\pi}{T} \delta(\omega - n\omega_0) = \frac{2\pi}{T} \sum_{n=-\infty}^{+\infty} \delta(\omega - n\omega_0)$$

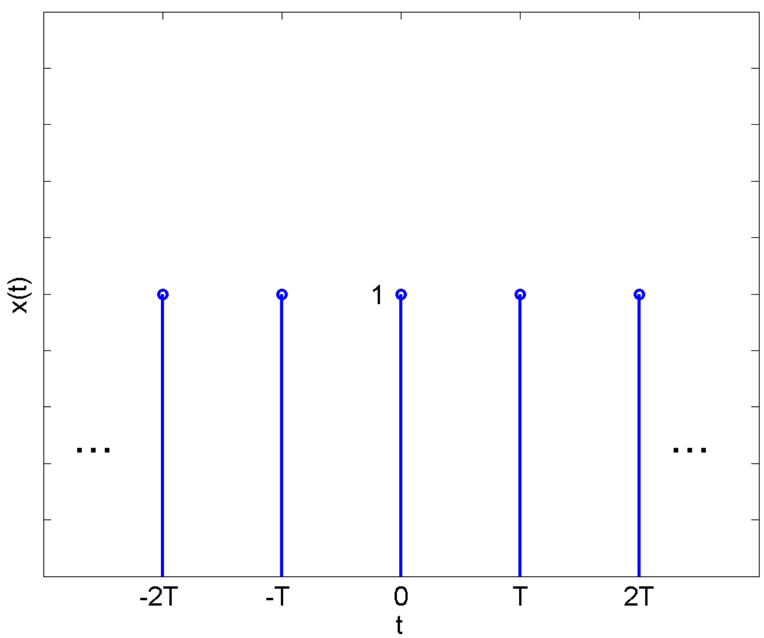
Train of Dirac Impulses

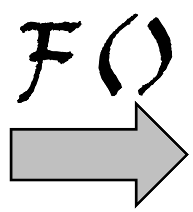
$$\omega_s = \frac{2\pi}{T}$$ Sampling angular frequency

$$x(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

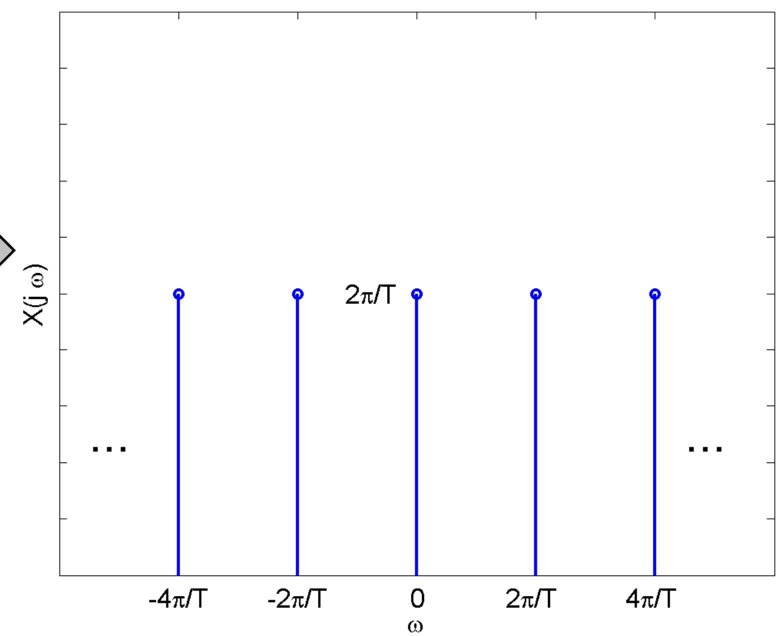
$$X(\omega) = \sum_{k=-\infty}^{\infty} \frac{2\pi}{T} \delta(\omega - k\omega_S)$$

Time domain





Frequency domain



Sampling in Frequency Domain

Time domain $x_p(t) = x(t)p(t)$

Frequency domain $X_p(\omega) = \frac{1}{2\pi} X(\omega) * P(\omega)$

$$P(\omega) = \frac{2\pi}{T} \sum_{k=-\infty}^{\infty} \delta(\omega - k\omega_S)$$

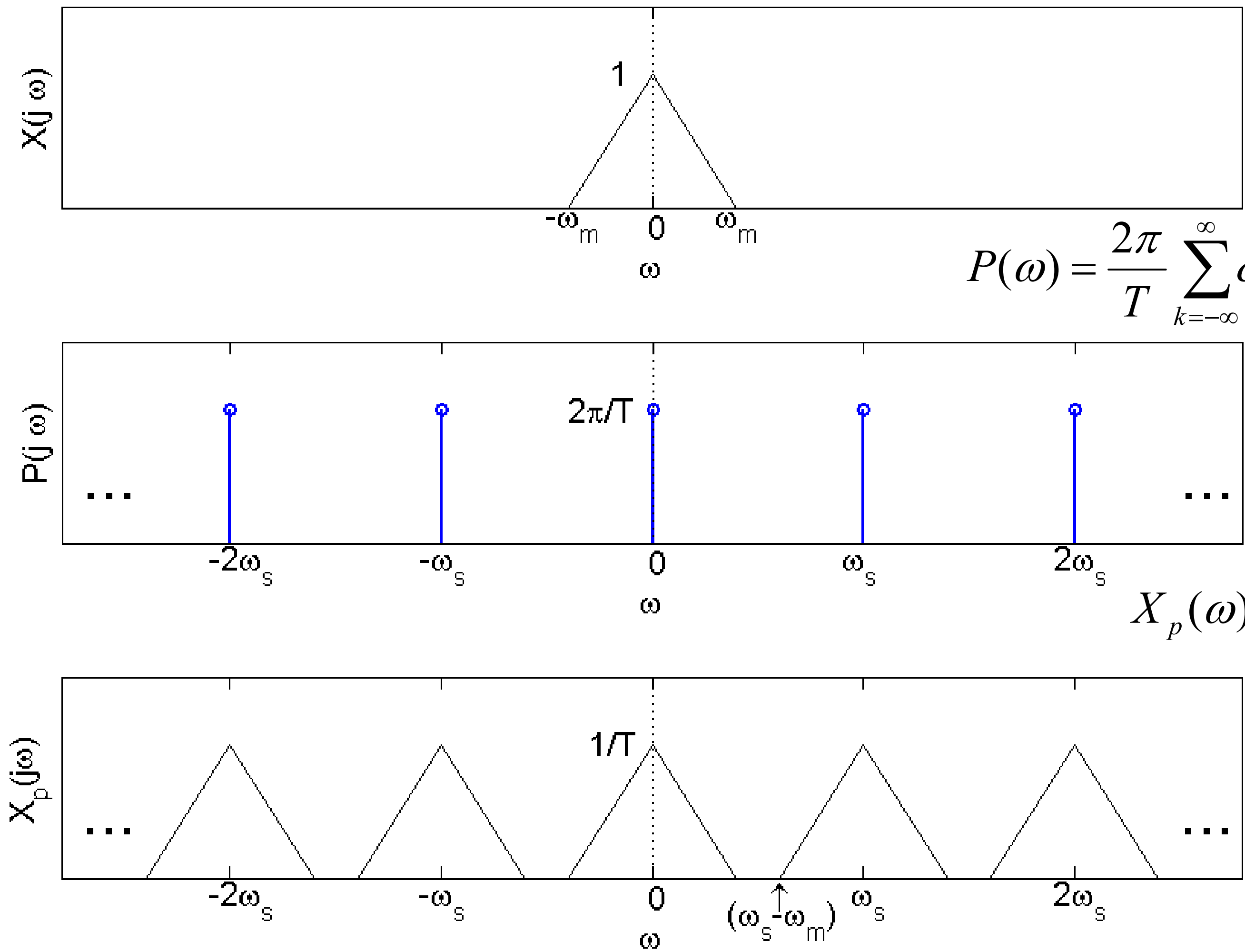
$$X_p(\omega) = \frac{1}{T} \sum_{k=-\infty}^{\infty} X(\omega) * \delta(\omega - k\omega_S)$$

$$= \frac{1}{T} \sum_{k=-\infty}^{\infty} X(\omega - k\omega_S)$$

Note: see also W2 ss. 45-47 as examples for this operation

Sampling a Band-Limited Signal

$X(\omega) \rightarrow$ spectrum of signal x(t)
with highest frequency < ω_m



$$P(\omega) = \frac{2\pi}{T} \sum_{k=-\infty}^{\infty} \delta(\omega - k\omega_S)$$

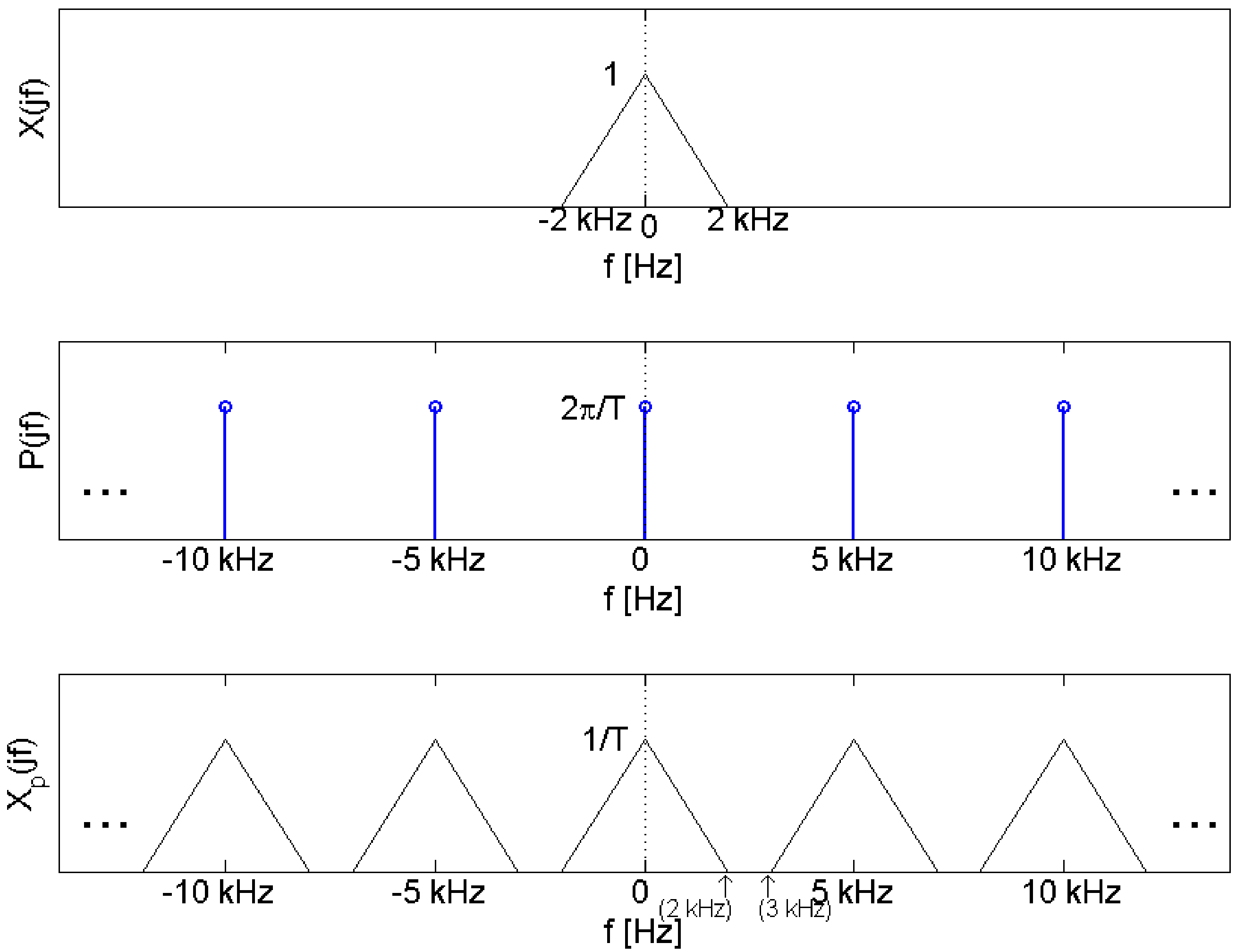
Angular sampling frequency
$\omega_s > 2\,\omega_m$

$$X_p(\omega) = \frac{1}{2\pi} X(\omega) * P(\omega)$$

$$= \frac{1}{T} \sum_{k=-\infty}^{\infty} X(\omega - k\omega_S)$$

Sampling a Band-Limited Signal

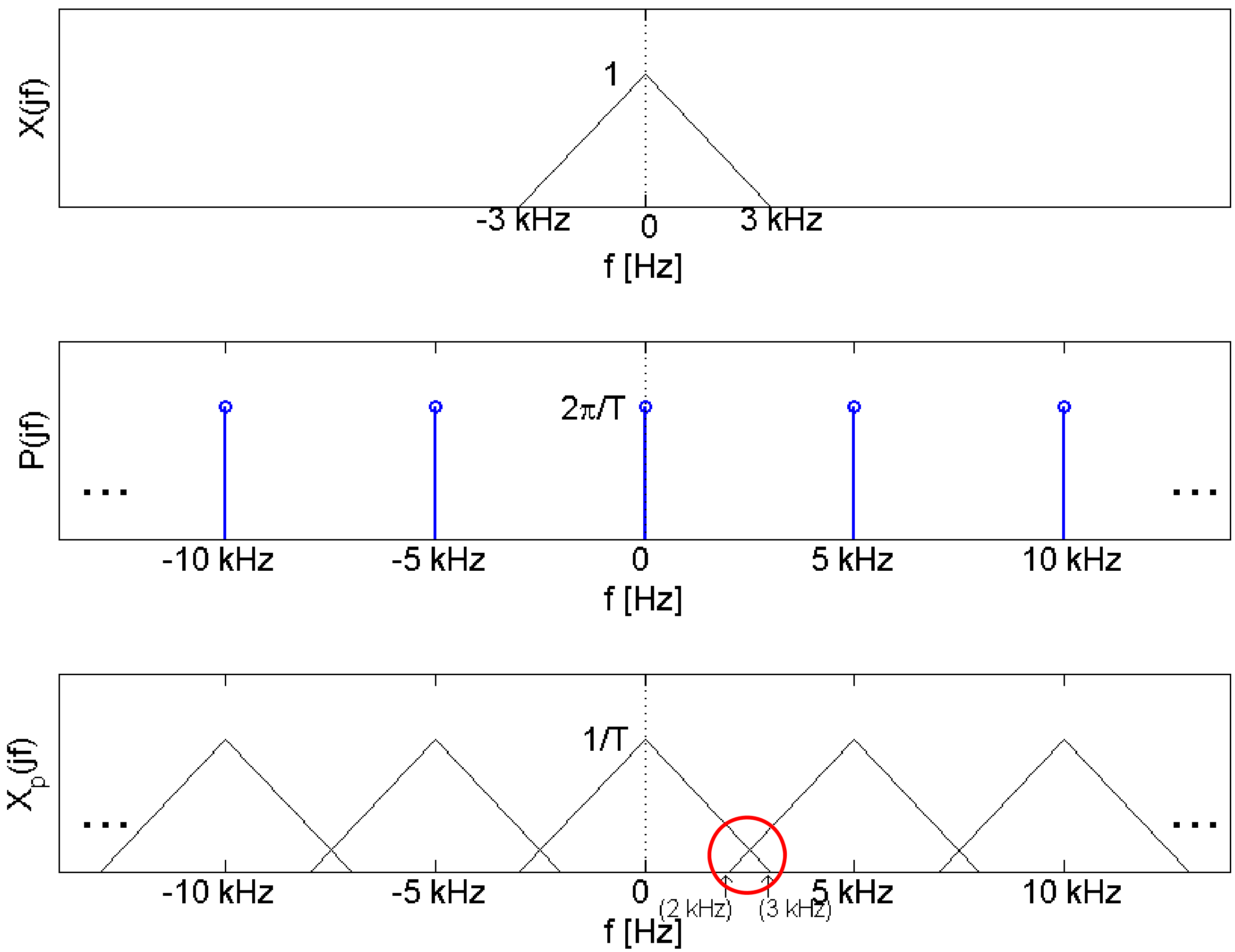
X(ξ) → spectrum of signal x(t)
with highest frequency < 2 kHz



Sampling frequency:
5 kHz > 2 * 2 kHz

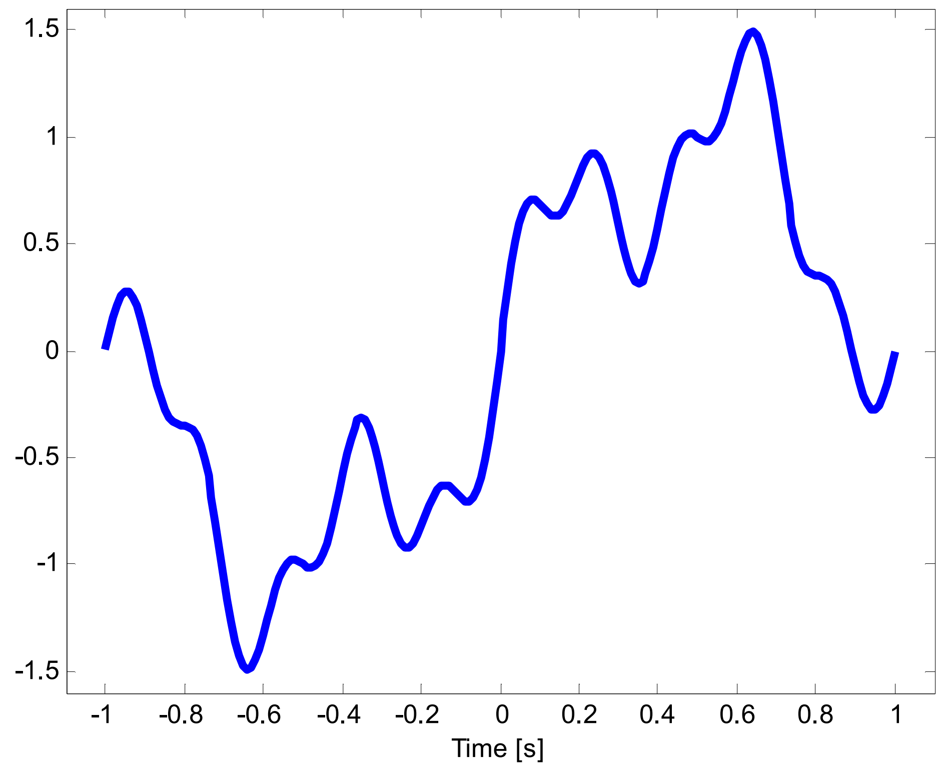
Sampling a Band-Limited Signal

X(ξ) → spectrum of signal x(t)
with highest frequency < 3 kHz



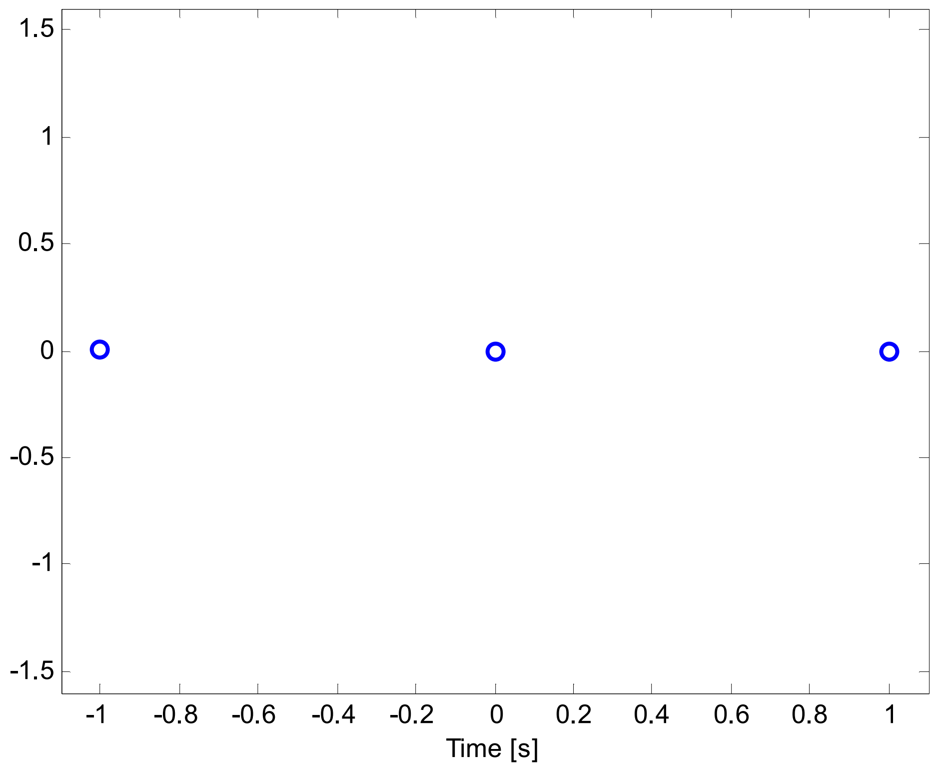
Sampling frequency:
5 kHz < 2 * 3 kHz

Original Signal



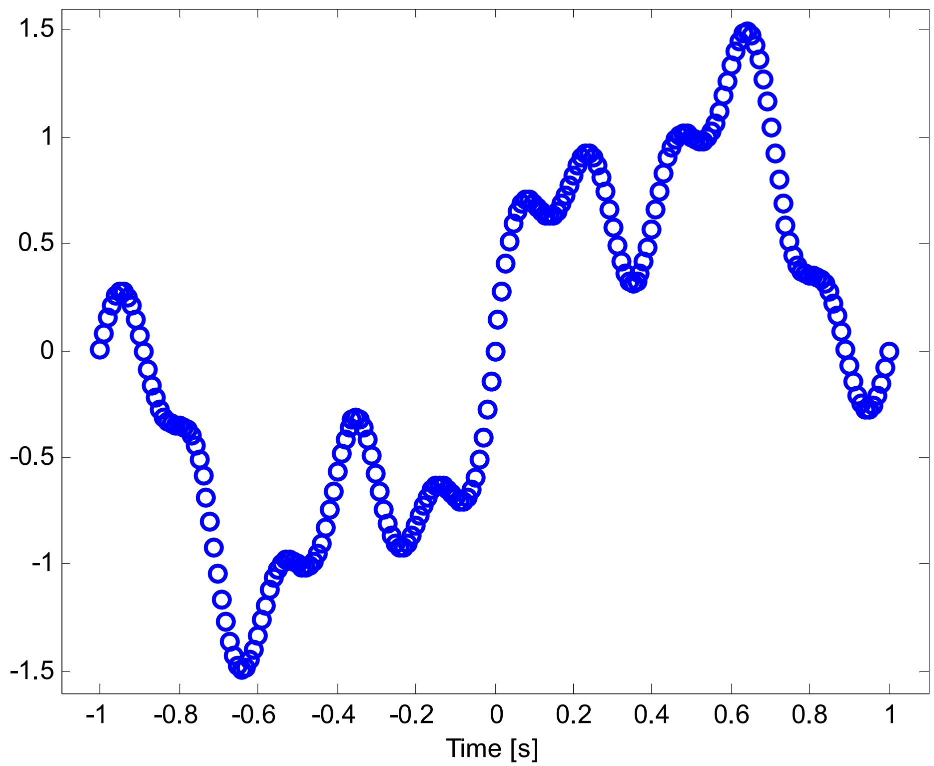
$$f(t) = \sin(2\pi t) + 0.4\sin(2\pi \cdot 2t) + 0.2\sin(2\pi \cdot 5t)$$

Too Few Samples (1Hz)



→ Data is lost

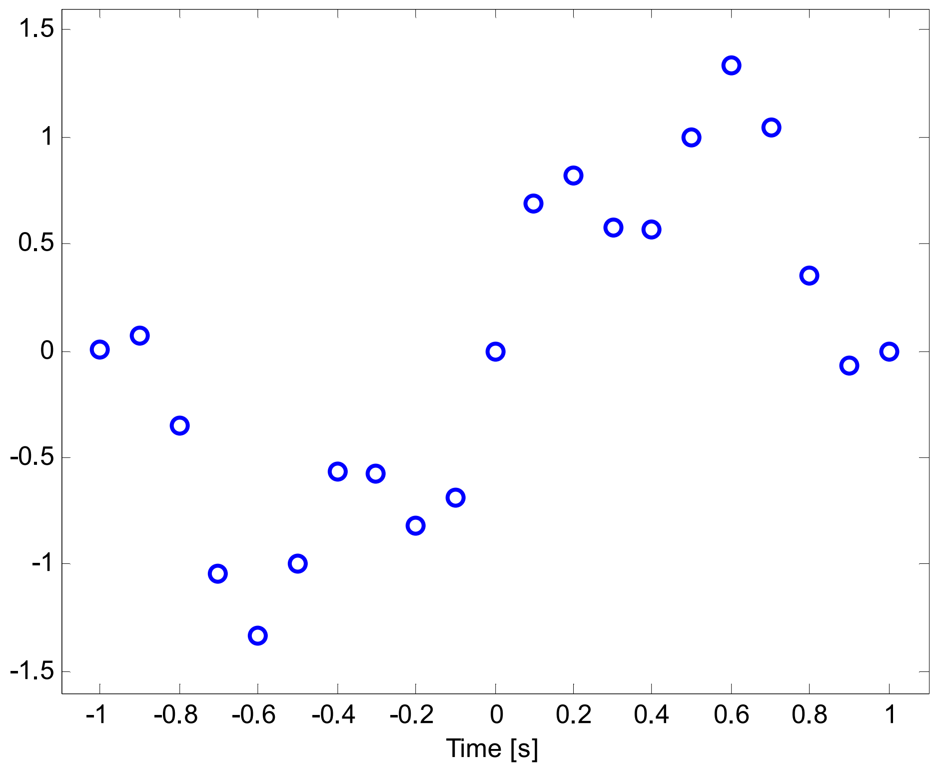
Too Many Samples (100 Hz)



→Redundant data

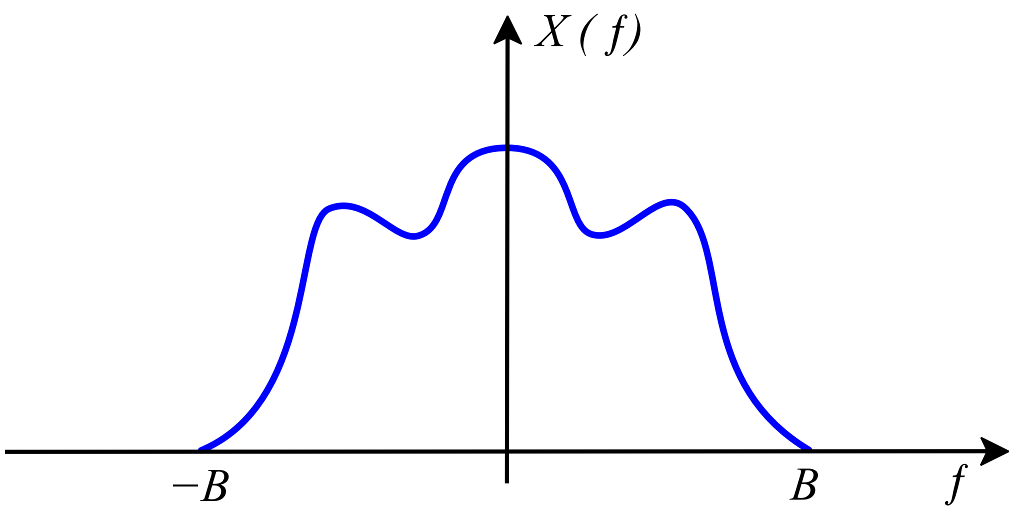
→Increase of data size

Minimal Possible Sampling (> 10 Hz)



Nyquist–Shannon Theorem

- If a function $x(t)$ contains no frequencies higher than B Hz, it is completely determined by giving its coordinates at a series of points spaced $1/(2B)$ seconds apart.
- Sampling frequency must be at least two times greater than the maximal signal frequency

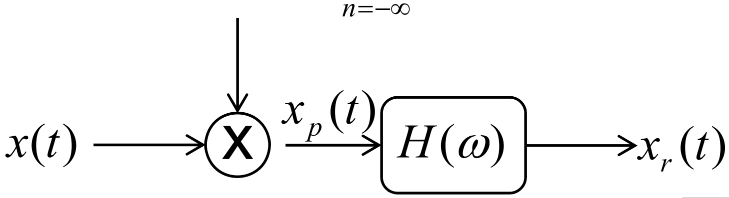


Sampling in Practice

- Sampling frequency two times greater than maximal frequency is the limit
- Example (parsimony principle applied): audio CD, sampling at 44.1 kHz since maximal hearable frequency is 20 kHz
- If affordable, try to use a sampling frequency 10 times greater than the maximal frequency (help all sorts of filtering and reconstruction processes)

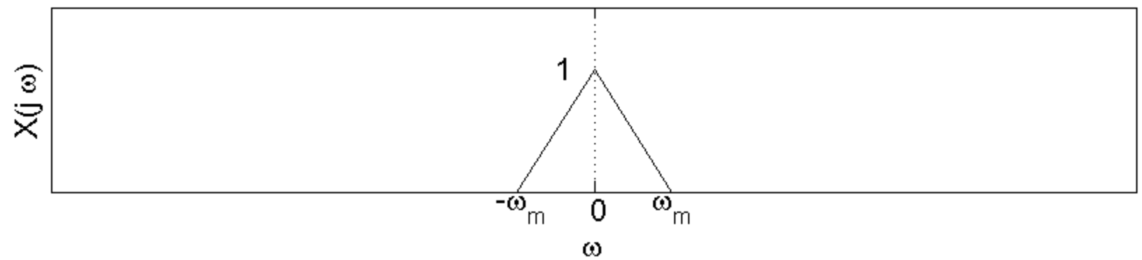
Signal Reconstruction

$$p(t) = \sum_{n=-\infty}^{+\infty} \delta(t - nT)$$



If there is no overlap between the shifted spectra the signal $x_r(t)$ can be perfectly reconstructed from x(t)

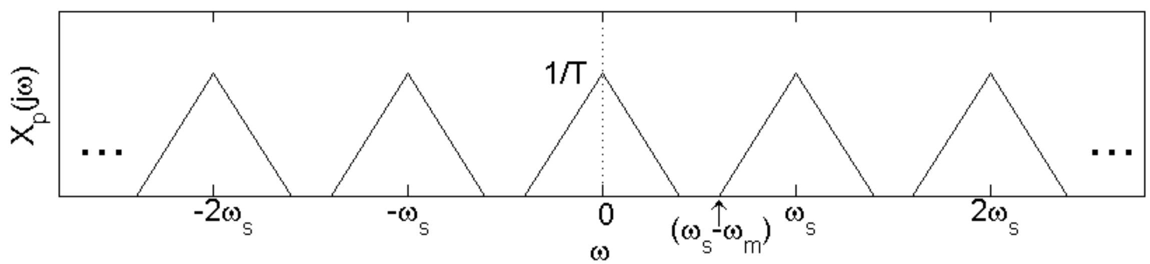
spectrum of original signal



spectrum of sampled signal

sampling angular frequency

$\omega_s > 2\ \omega_m$

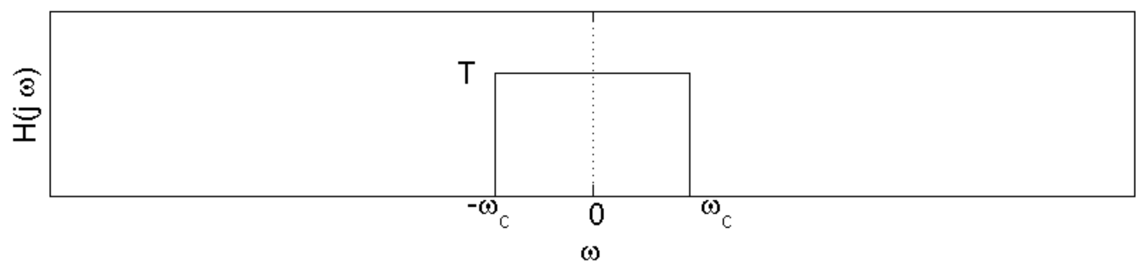


filtering

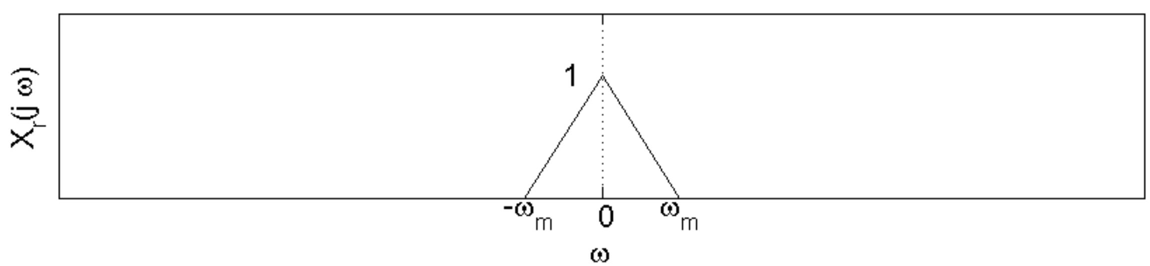
filter cut-off angular frequency

$\omega_m < \omega_c < (\omega_s - \omega_m)$

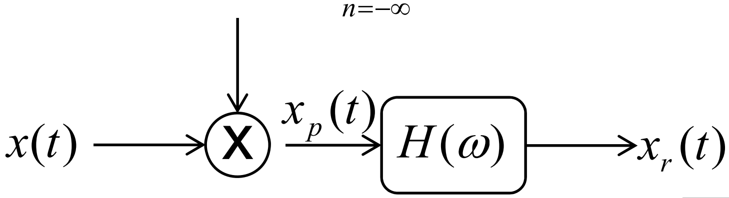
$$X_r(\omega) = X_p(\omega) H(\omega)$$



spectrum of reconstructed signal

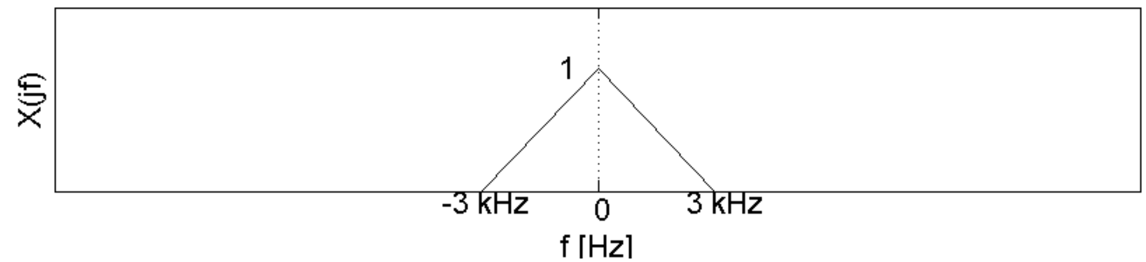


$$p(t)=\sum_{n=-\infty}^{+\infty}\delta(t-nT)$$



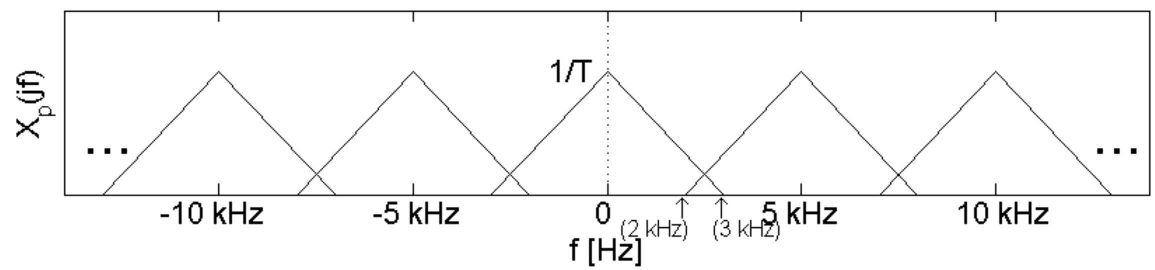
If there is overlap between the shifted spectra the signal $x_r(t)$ cannot be perfectly reconstructed from x(t)

spectrum of original signal



spectrum of sampled signal

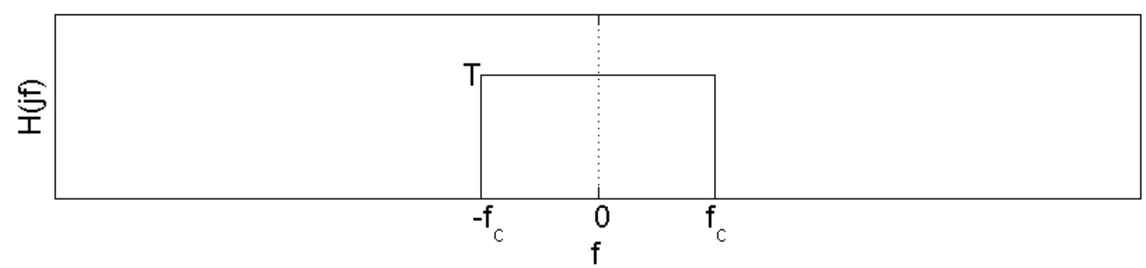
sampling frequency
$f_s < 2\ f_m$



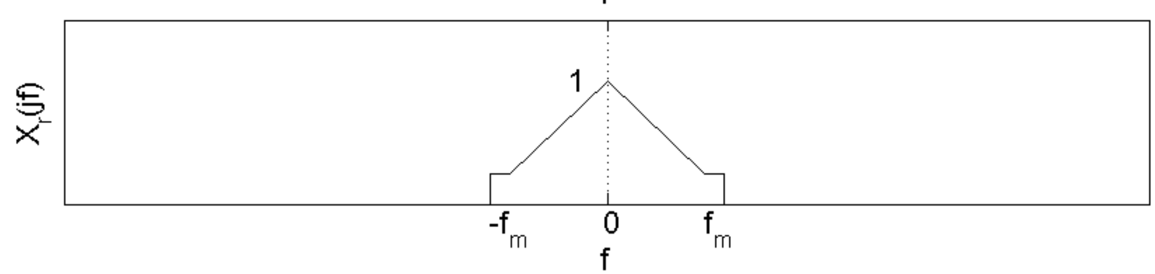
filtering

filter cut-off frequency f_c
$f_m < f_c < (f_s - f_m)$

$$X_r(\omega)=X_p(\omega)H(\omega)$$

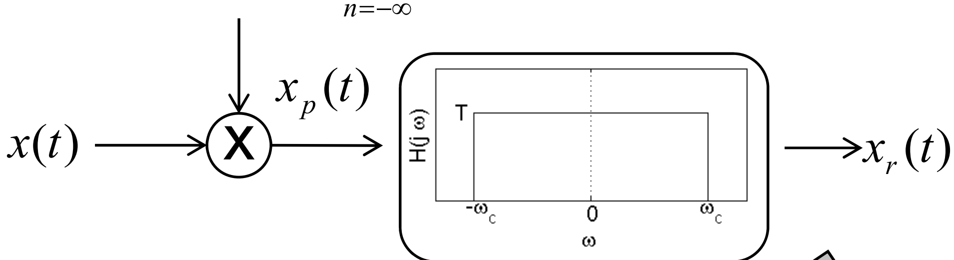


spectrum of reconstructed signal



Time Domain Interpretation of Signal Reconstruction

$$p(t)=\sum_{n=-\infty}^{+\infty}\delta(t-nT)$$



$\omega_m < \omega_c < (\omega_s - \omega_m)$

- Multiplication with a Low-Pass Filter (LPF) in the frequency domain
- The LPF interpolates the samples assuming x(t) contains no energy at frequencies > ω_c (ω_c =cutoff angular frequency)

$\mathcal{F}^{-1}()$

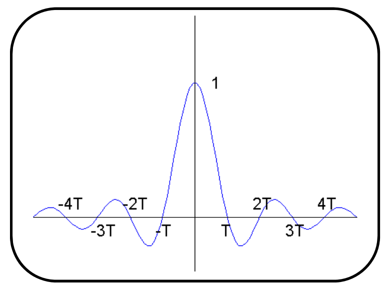
$$x_r(t)=x_p(t)*h(t)$$

$$=\left(\sum_{n=-\infty}^{\infty}x(nT)\delta(t-nT)\right)*h(t)$$

$$=\sum_{n=-\infty}^{\infty}x(nT)h(t-nT)$$

$$=\sum_{n=-\infty}^{\infty}x(nT)\frac{T\sin(\omega_c(t-nT))}{\pi(t-nT)}$$

with $h(t)=\dfrac{T\sin(\omega_c t)}{\pi t}$



Note: see also W2 ss. 45-47 as examples for this operation

Signal Reconstruction in Practice

1. **Whittaker-Shannon interpolation** (band-limited interpolation):

- Signal has to be band limited
(i.e. Fourier transform for frequencies greater than B equal 0)
- The sampling rate must exceed twice the bandwidth, 2B, i.e. $f_s > 2B$

Assume: $\omega_c = \frac{\omega_S}{2} = \frac{2\pi}{2T} = \frac{\pi}{T}$ in s. 28, $\text{sinc}(x) = \frac{\sin(\pi x)}{\pi x}$ Normalized sinc function

$$x_r(t) = \sum_{n=-\infty}^{\infty} x(nT) \cdot \text{sinc}\left(\frac{t - nT}{T}\right)$$

$$x_r(t) = \left(\sum_{n=-\infty}^{\infty} x(nT) \cdot \delta(t - nT)\right) * \text{sinc}\left(\frac{t}{T}\right)$$

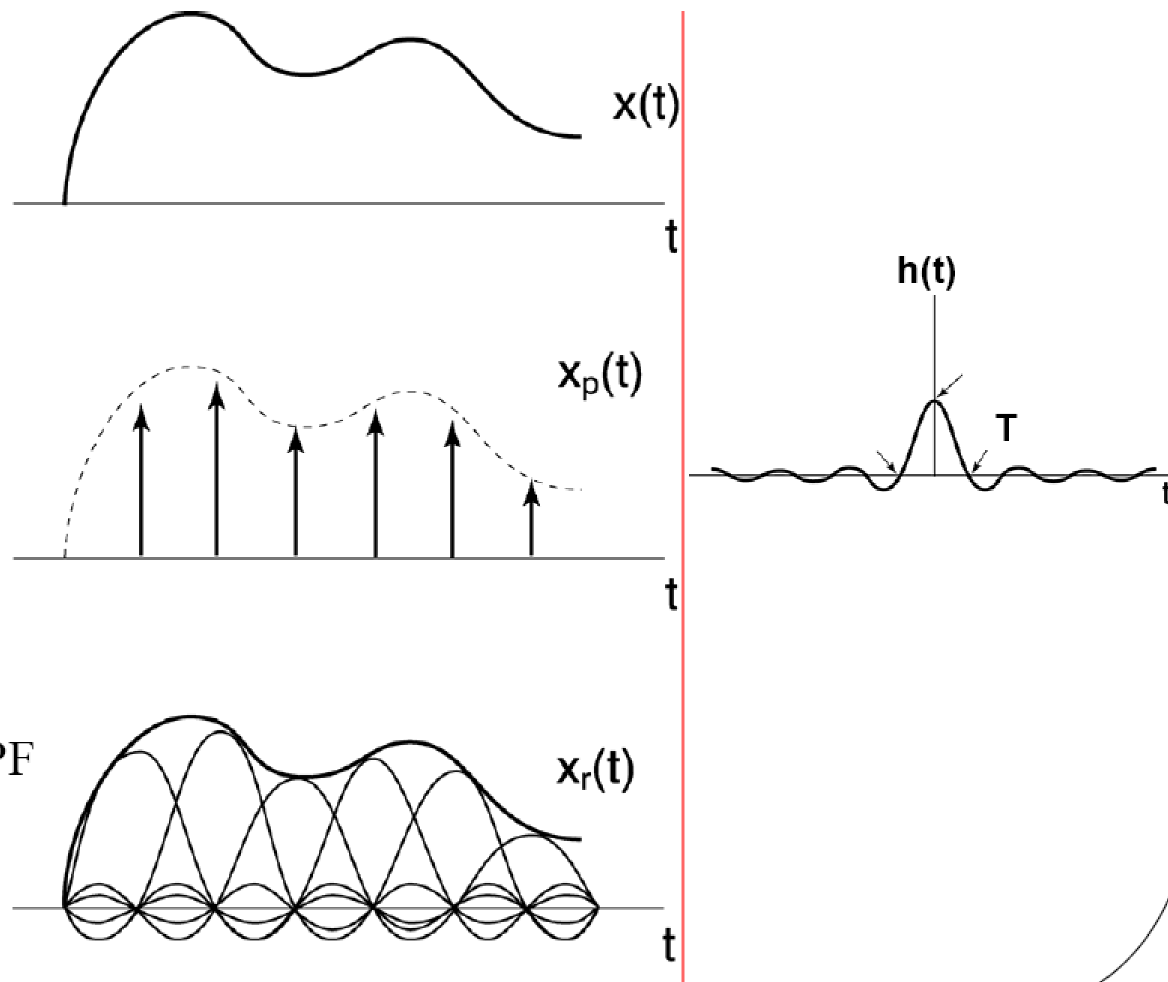
(Alternative equivalent formulation)

Graphic Illustration of Time-Domain Interpolation

Original
CT signal

After sampling

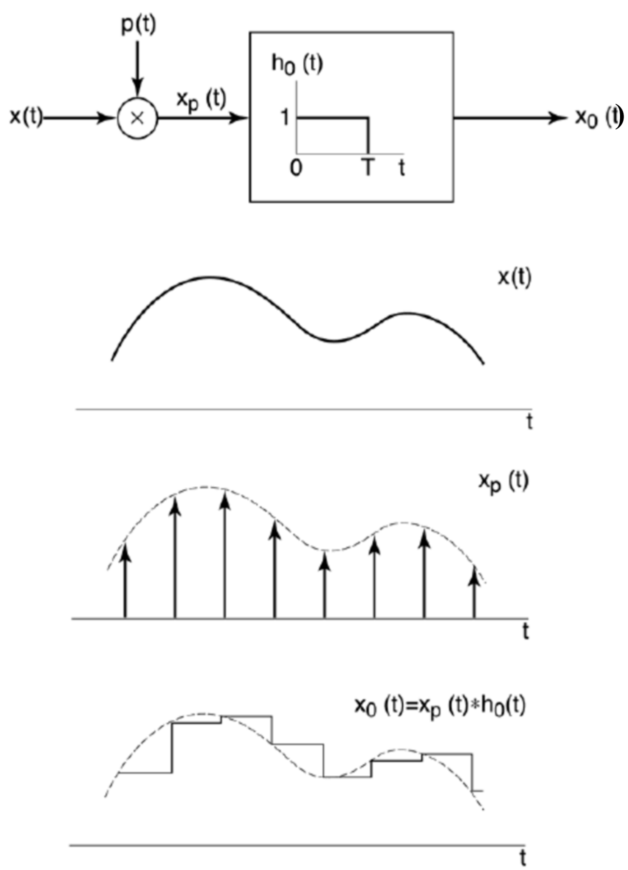
After passing the LPF
(Low Pass Filter)



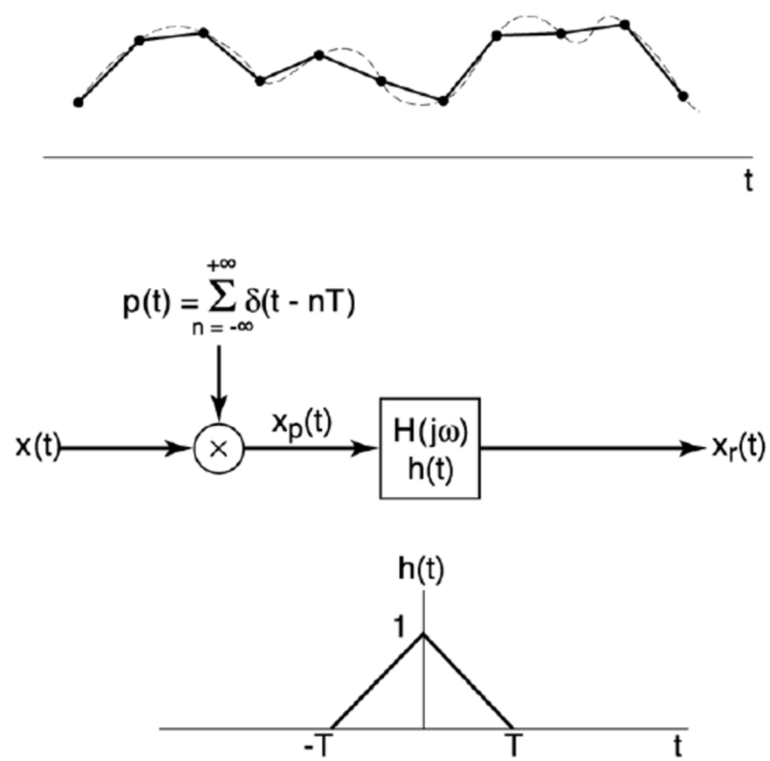
From Prof. A. S. Willsky, Signals and Systems course

Signal Reconstruction in Practice

2. Zero-order hold



3. First-order hold (linear interpolation)



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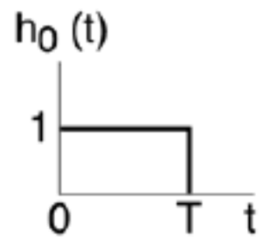
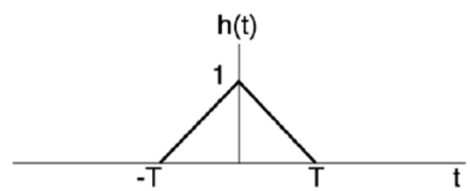
Reconstruction Summary – Time Domain

The reconstructed signal $x_r(t)$ is obtained through a convolution between the sampled signal $x_p(t)$ with period T and one of the following three interpolation functions h(t).

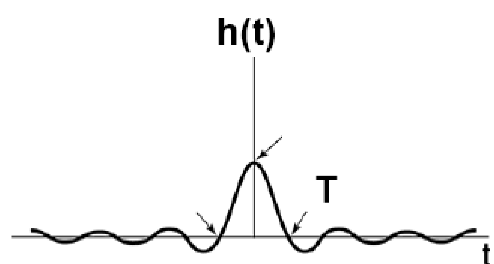
1. Zero-order hold (ZOH)

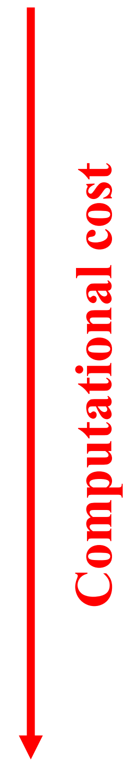


2. First-order hold (FOH)



3. Whittaker-Shannon





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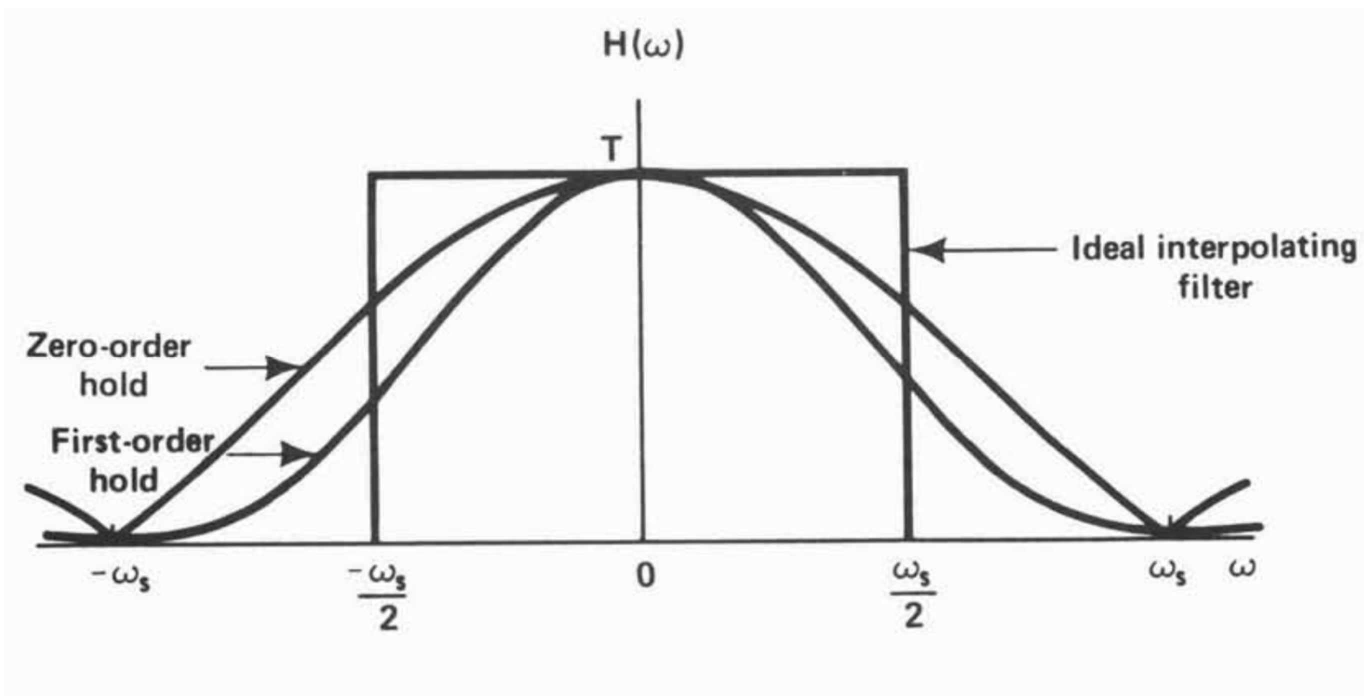
Reconstruction Summary – Frequency Domain

The spectrum of the reconstructed signal $X_r(\omega)$ is obtained through a multiplication between the spectrum of the sampled signal $X_p(\omega)$ with angular sampling frequency ω_s and one of the following three low-pass filters.



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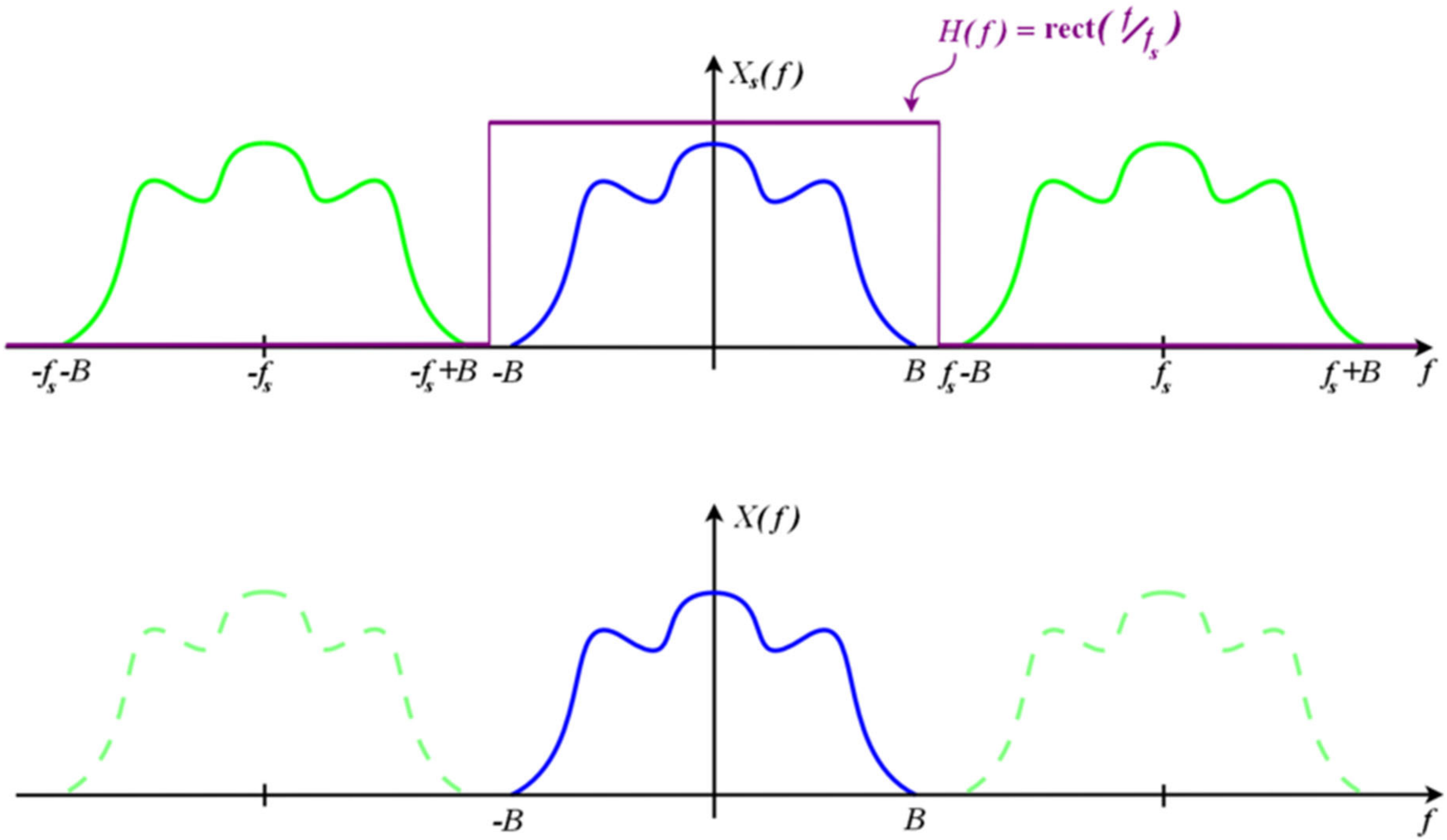
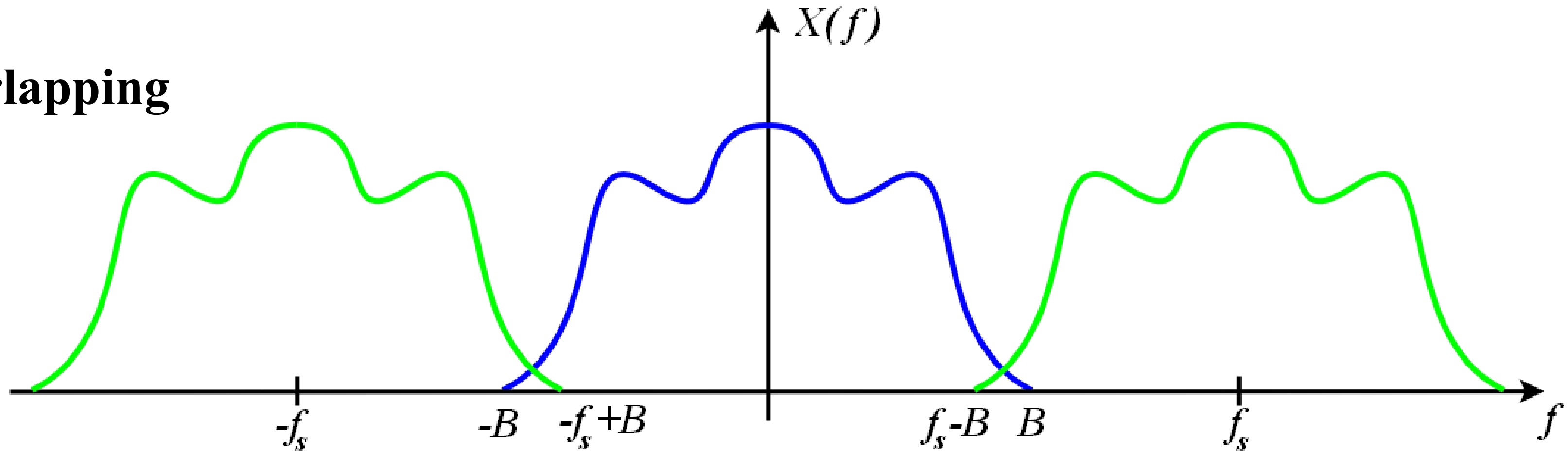
Aliasing

No Problems in Reconstruction

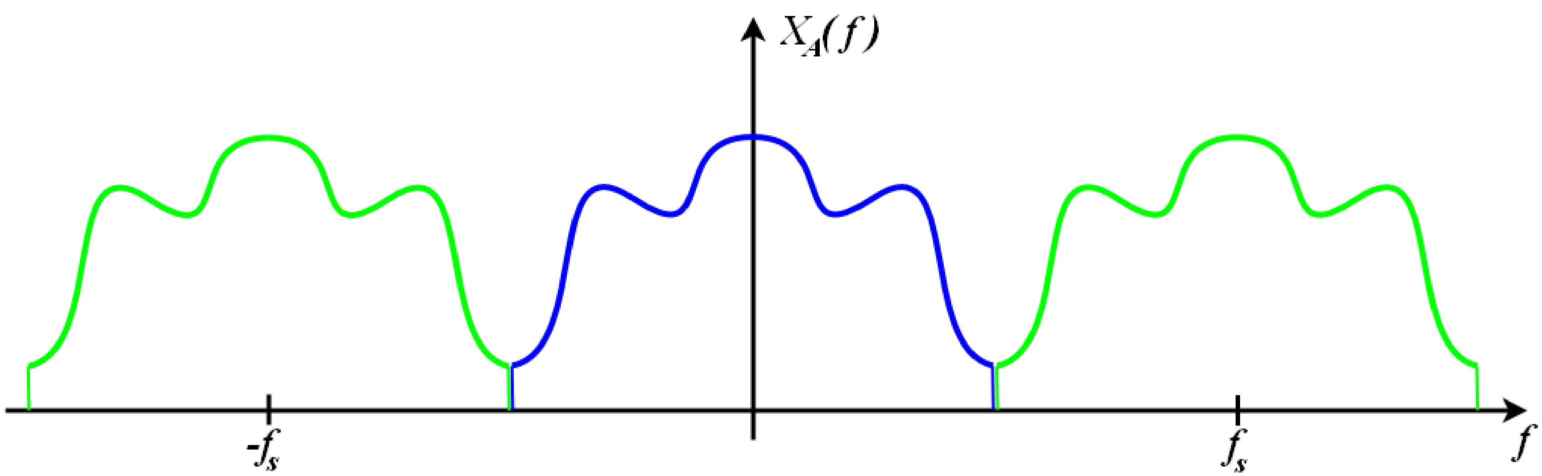


Reconstruction Problems

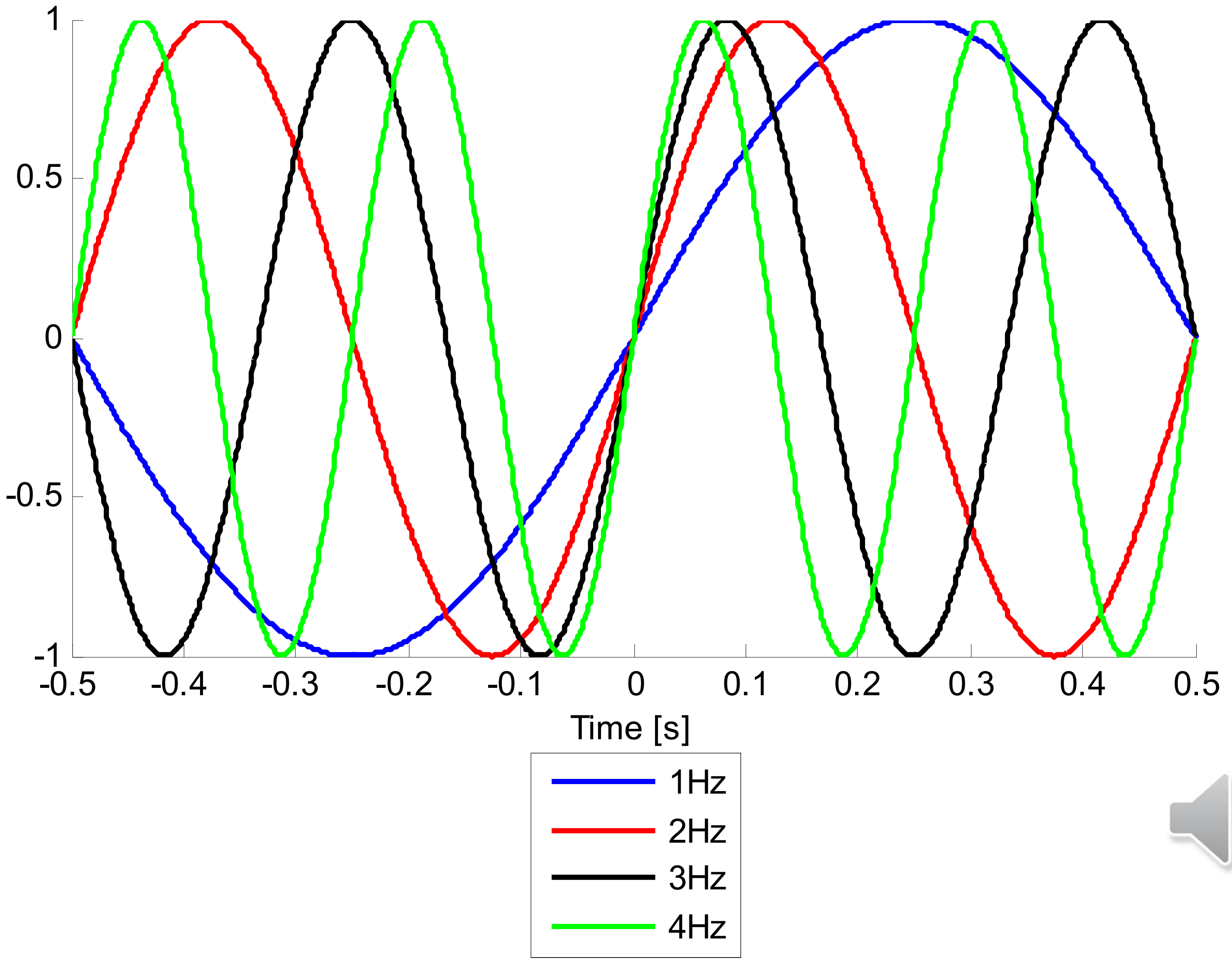
Overlapping



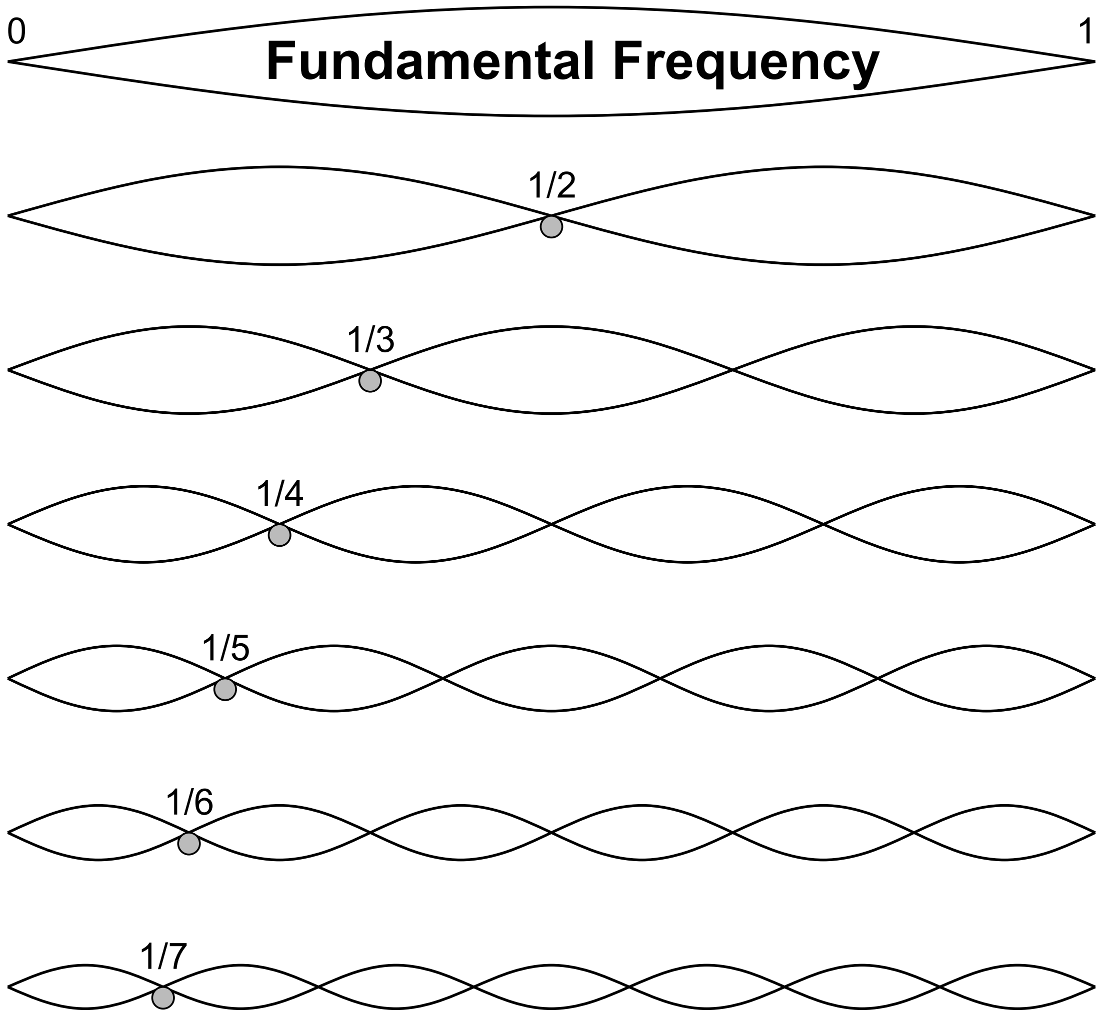
Alias



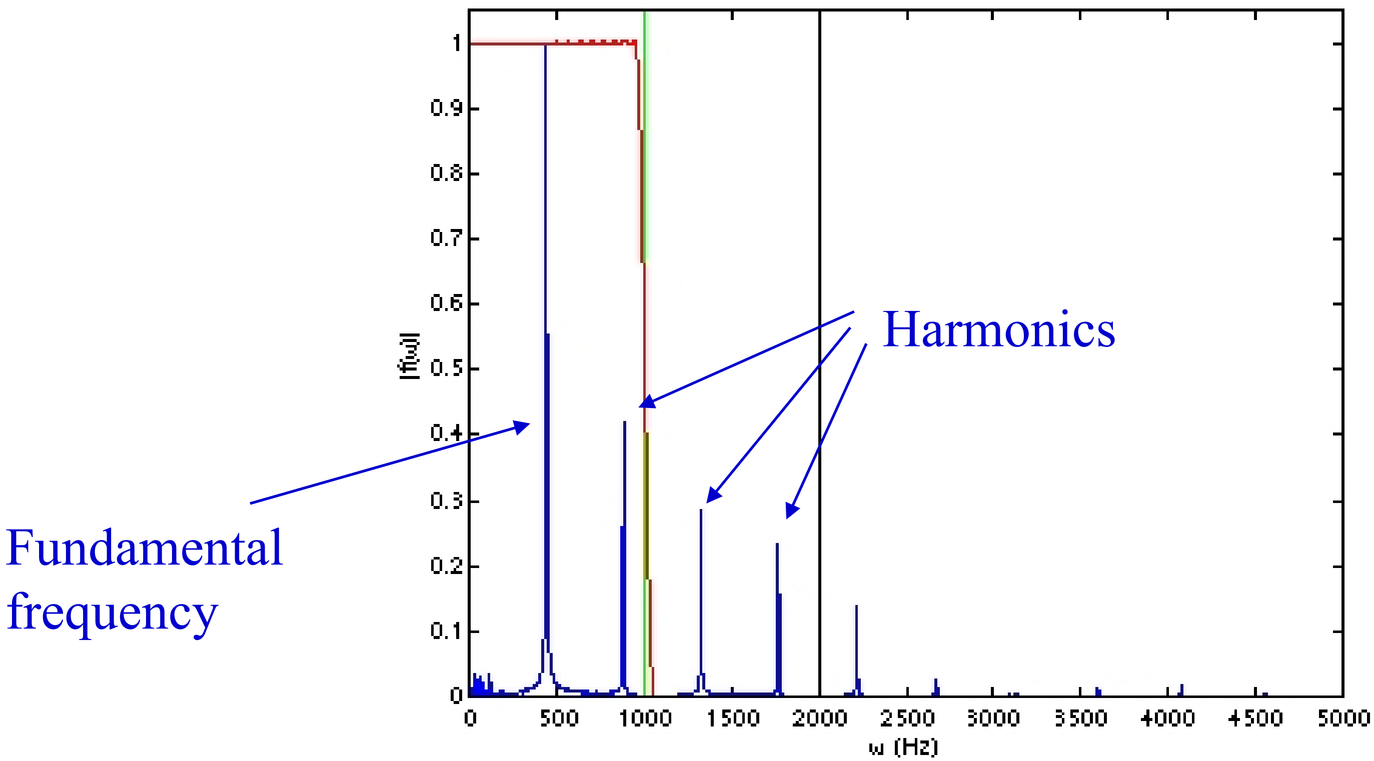
Harmonics



Harmonics



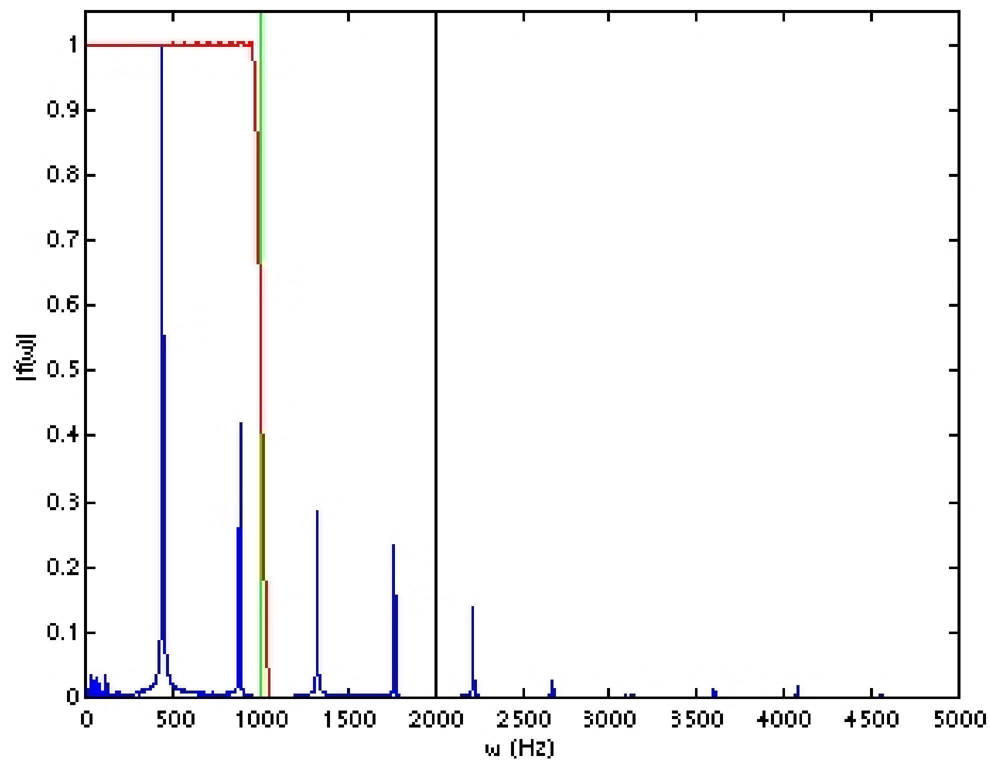
La-Tone (440 Hz) sampled at 44.1 kHz (CD standard)



Rem: s. 6, W2, max audible signals by humans 20 kHz

La-Tone (440 Hz) sampled at 2 kHz

Reduced sampling frequency: 2 kHz

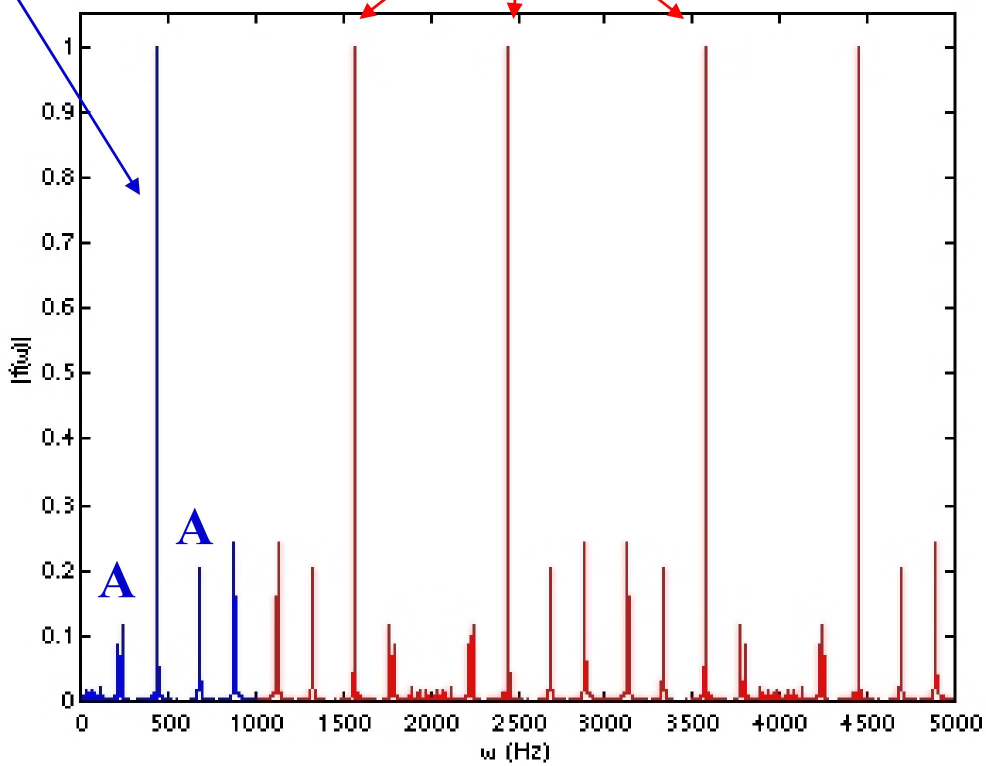


EPFL

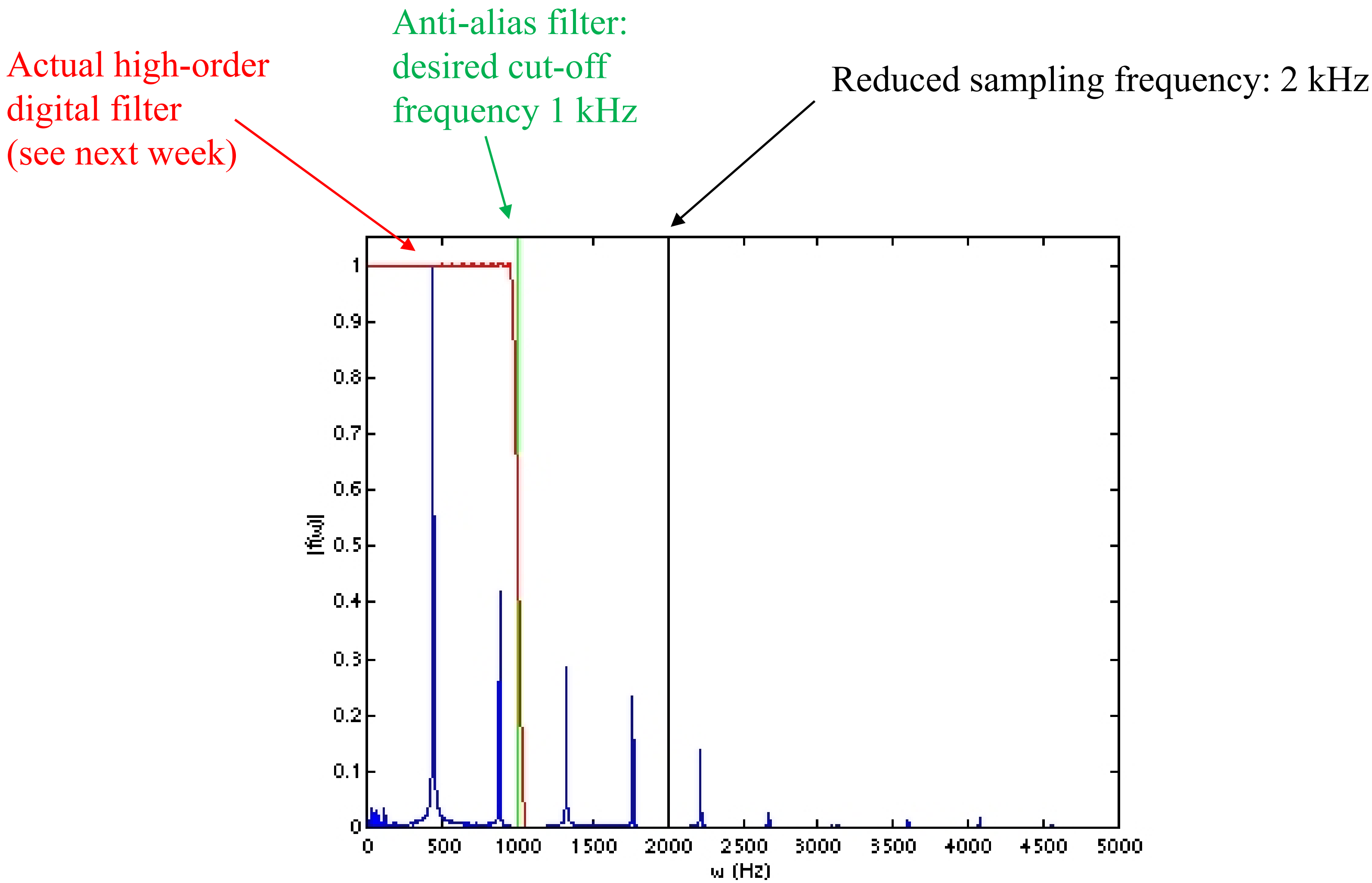
La-Tone (440 Hz) sampled at 2 kHz without filtering

Original signal with aliasing effect (A)

Repeated & shifted spectrum through sampling



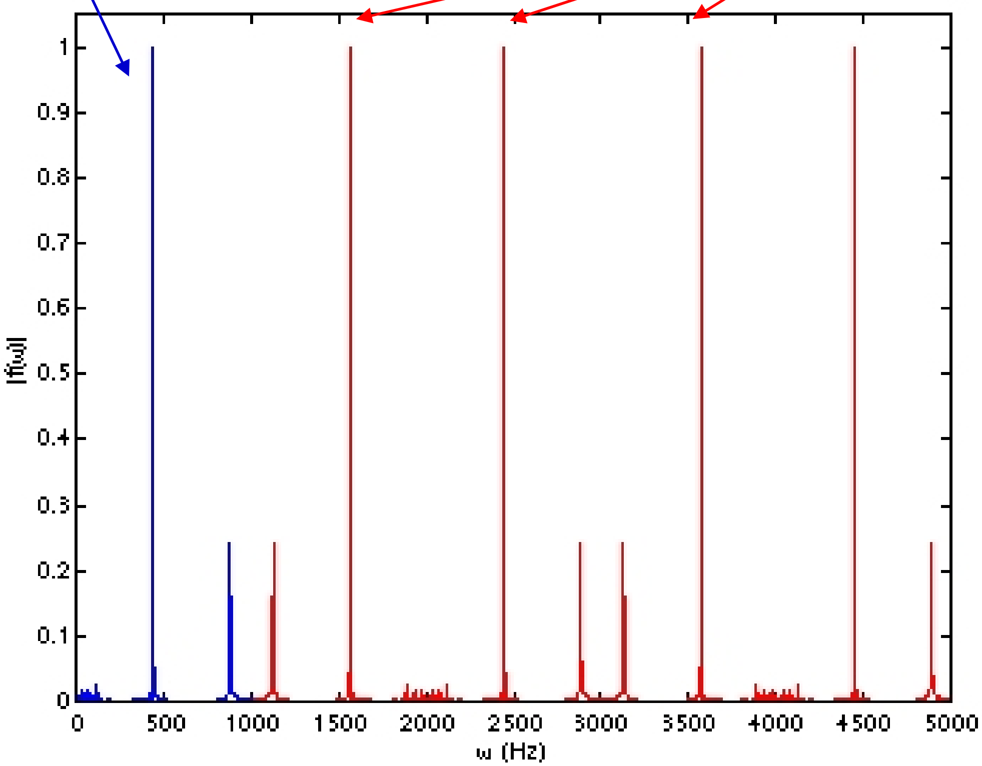
La-Tone (440 Hz) sampled at 2 kHz



La-Tone (440 Hz) sampled at 2 kHz filtered at 1 kHz

Original signal without aliases through anti-alias filtering

Repeated & shifted spectrum through sampling will be eliminated through low-pass filtering at signal reconstruction stage; however, harmonics above 1 kHz also cut by the anti-alias filter

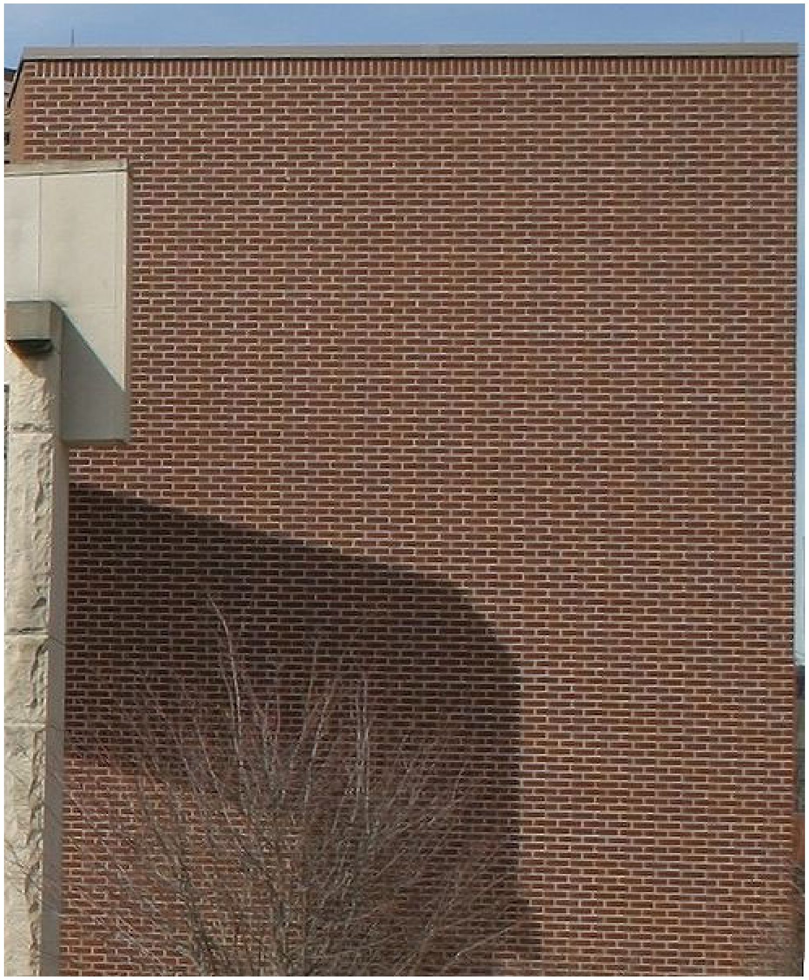


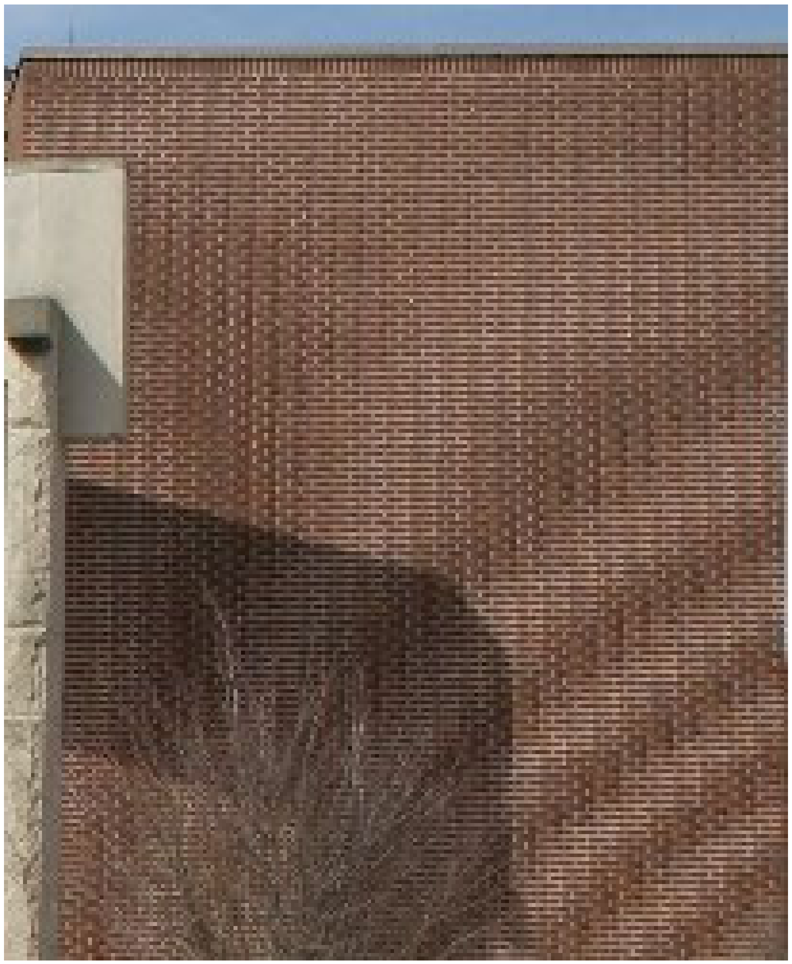
Aliasing Audio Examples

Original sound | **Aliases 4 kHz** | **Correct sampling 4 kHz**

Note: for instance cymbals (8-16 kHz) fully cut off, no matter what

Moiré Pattern





Aliasing Video Examples

https://www.youtube.com/watch?v=R-IVw8OKjvQ

https://www.youtube.com/watch?v=jHS9JGkEOmA

Conclusion

Take Home Messages

- Sampling: multiplication of the signal with a periodic train of Dirac impulses results in a repeated shifted spectrum in the frequency domain
- Multiple signal reconstructions algorithms are available, more or less computationally expensive; they can all be represented by low-pass filters in the frequency domain
- Aliasing: higher frequency "folded back" on the original spectrum -> prevent proper signal reconstruction
- When you sample a signal …
  - Make sure you know what the maximum frequency f_{max} is or enforce it through an anti-alias low-pass filter
  - Make sure you sample at $f_s > 2\ f_{max}$ (Nyquist-Shannon theorem)

Additional Literature – Week 3

- **Books:**
- J. H. McClellan, R. W. Schafer, M. A. Yoder "DSP First: A Multimedia Approach", Prentice Hall, 1999.
- A. Oppenheim and A. S. Willsky with S. Nawab, "Signals and Systems", Prentice Hall, 1997.