EPFL

Disal

## Signals, Instruments, and Systems – W2

# An Introduction to Signal Processing – Fourier Series, Fourier Transforms, and Convolution

# Outline

- Some typical signals and values
- Fourier series
- Continuous and discrete Fourier transforms
- Continuous and discrete convolution

# Typical Waves and Signals

# Waves

"A wave is a disturbance that propagates through space and time, usually with transference of energy."

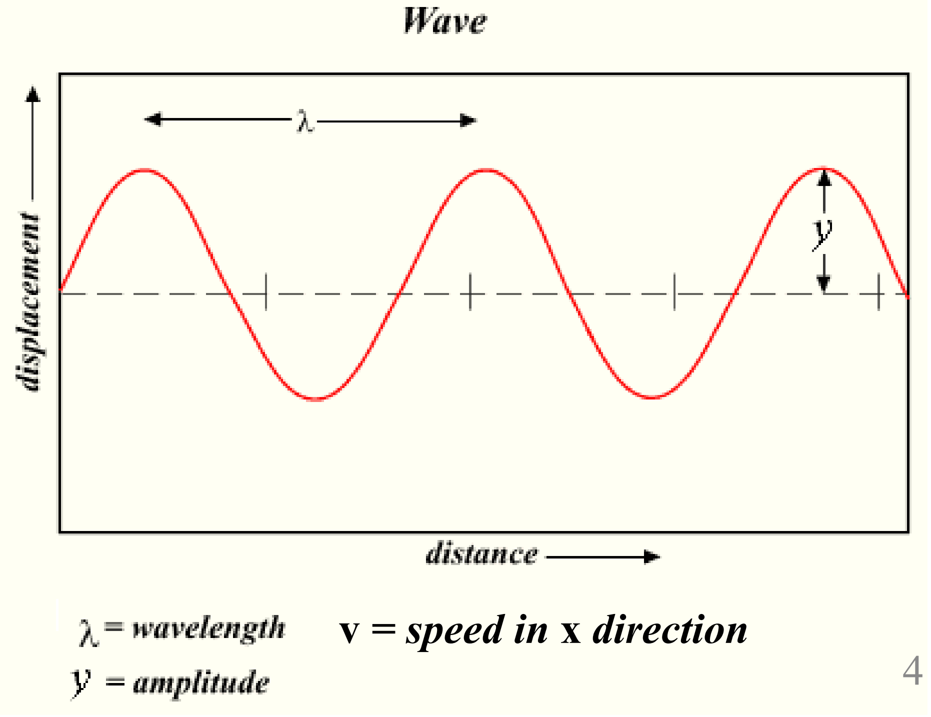
Wave function:

$$f(x,t) = \gamma \sin(kx - \omega t)$$

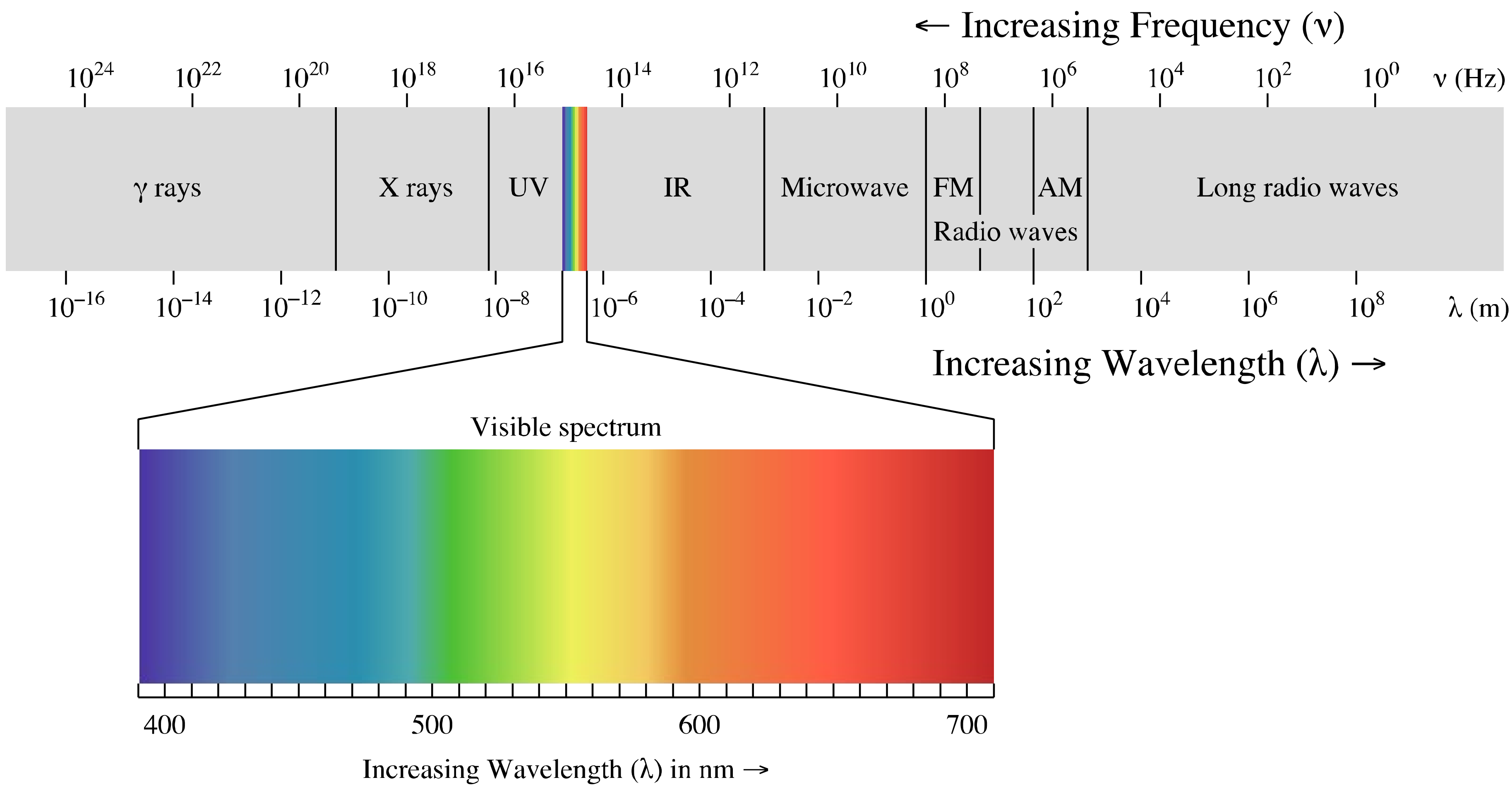
$$k = \frac{2\pi}{\lambda}$$ "wave number"

$$\omega = 2\pi f$$ "angular frequency"

$$\lambda = \frac{\mathrm{v}}{\mathrm{f}}$$ "wave length"



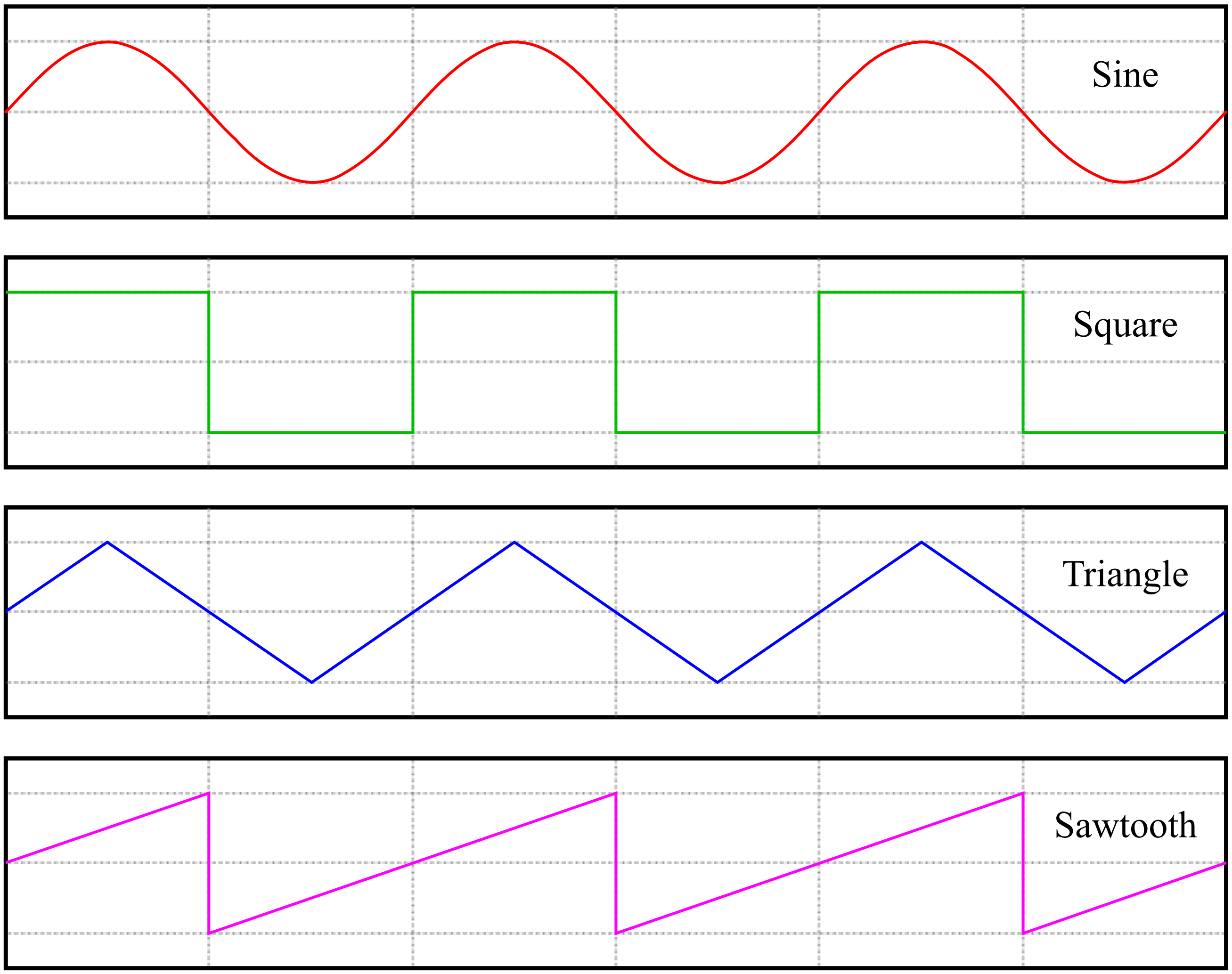
# Electromagnetic Spectrum



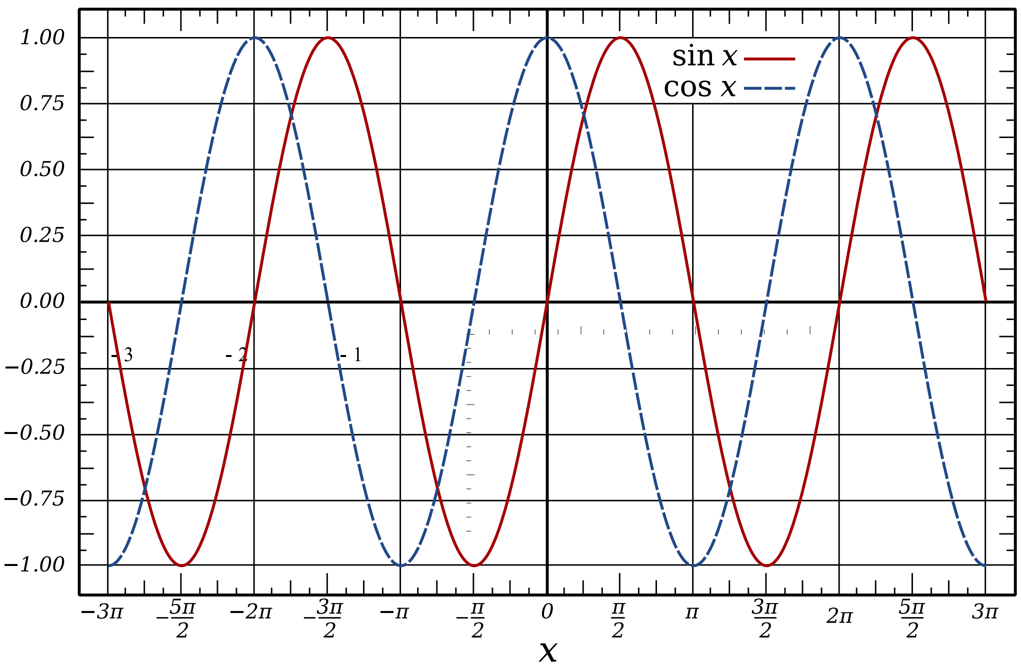
# Common Frequencies

- Car motor: ~50 Hz (3000 rpm)
- Power lines: 50 Hz
- Ear: 20 Hz – 20 kHz
- Ultrasound: 20 kHz – 200 MHz
- Medical sonograph: ~2-18 MHz
- Watch quartz: 32 kHz
- CPU: 2-3 GHz
- GSM: 0.9/1.8 GHz

# Standard Waveforms



# Sinusoid (Sine Wave) in Time




$A$ : amplitude

$\omega$ : angular frequency

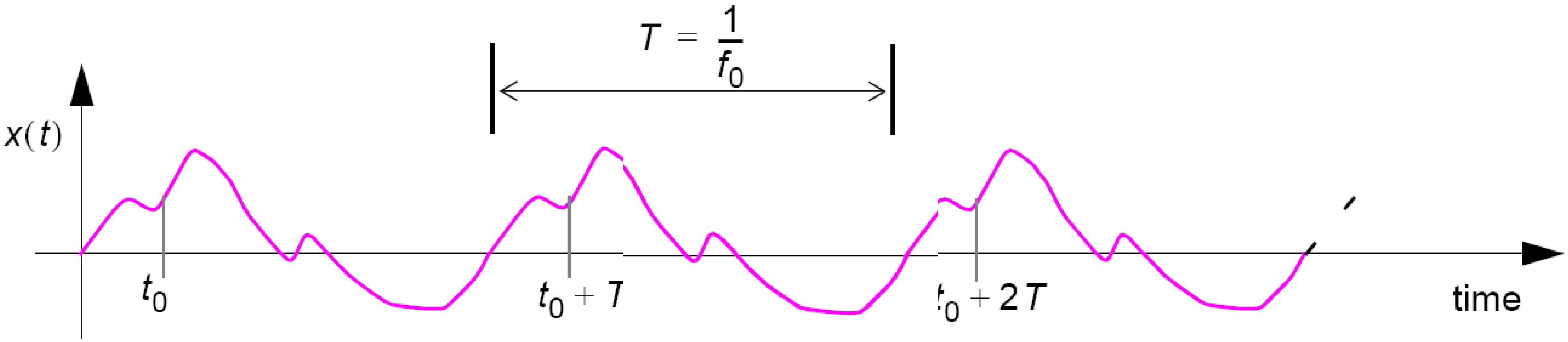
$\theta$ : phase

$$y(t) = A \cdot \sin(\omega t + \theta)$$

# Fourier Series

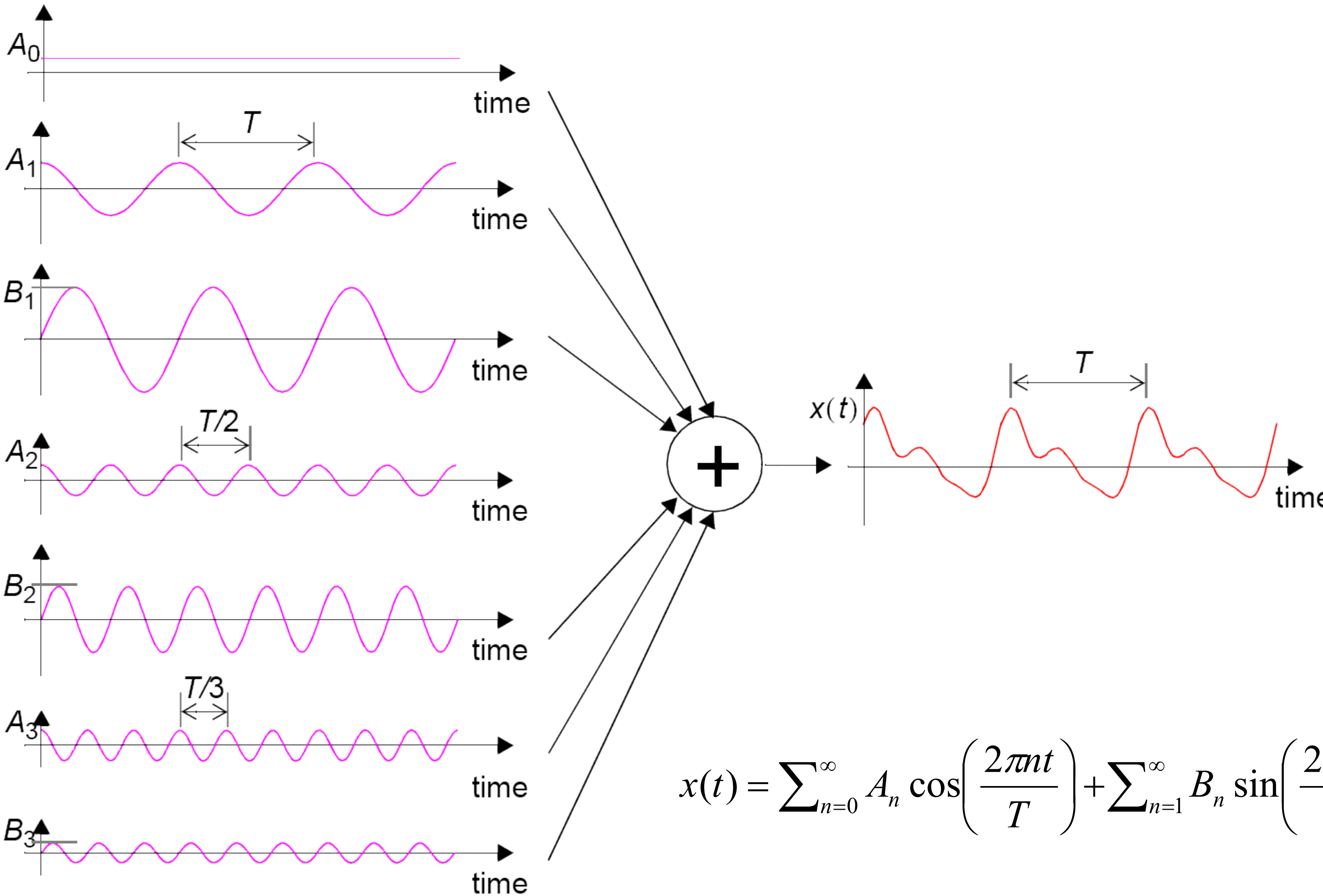
# Fourier Series

- Joseph Fourier (1768-1830) proposed that any periodic function can be decomposed into a sum of simple oscillating functions, namely sines and cosines.




"Any periodic function …"

from: Robert W. Stewart, Daniel García-Alís, "Concise DSP Tutorial"




$$x(t) = \sum_{n=0}^{\infty} A_n \cos\left(\frac{2\pi nt}{T}\right) + \sum_{n=1}^{\infty} B_n \sin\left(\frac{2\pi nt}{T}\right)$$

"…can be decomposed into a sum of sines and cosines"

from: Robert W. Stewart, Daniel García-Alís, "Concise DSP Tutorial"

# Fourier Series

$$f(t) = \sum_{n=0}^{\infty} A_n \cos\left(\frac{2\pi nt}{T}\right) + \sum_{n=1}^{\infty} B_n \sin\left(\frac{2\pi nt}{T}\right)$$

- Suppose a 2π periodic function f(t) integrable on [−π, π]
- The corresponding Fourier coefficients are:

$$A_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos(nt)\,dt, \quad n \geq 0$$

$$B_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin(nt)\,dt, \quad n \geq 1$$

# Complex Numbers Review

Rectangular form : $C_n = \mathrm{Re}(C_n) + i\,\mathrm{Im}(C_n)$

Euler's formula : $e^{i\varphi} = \cos\varphi + i\sin\varphi$

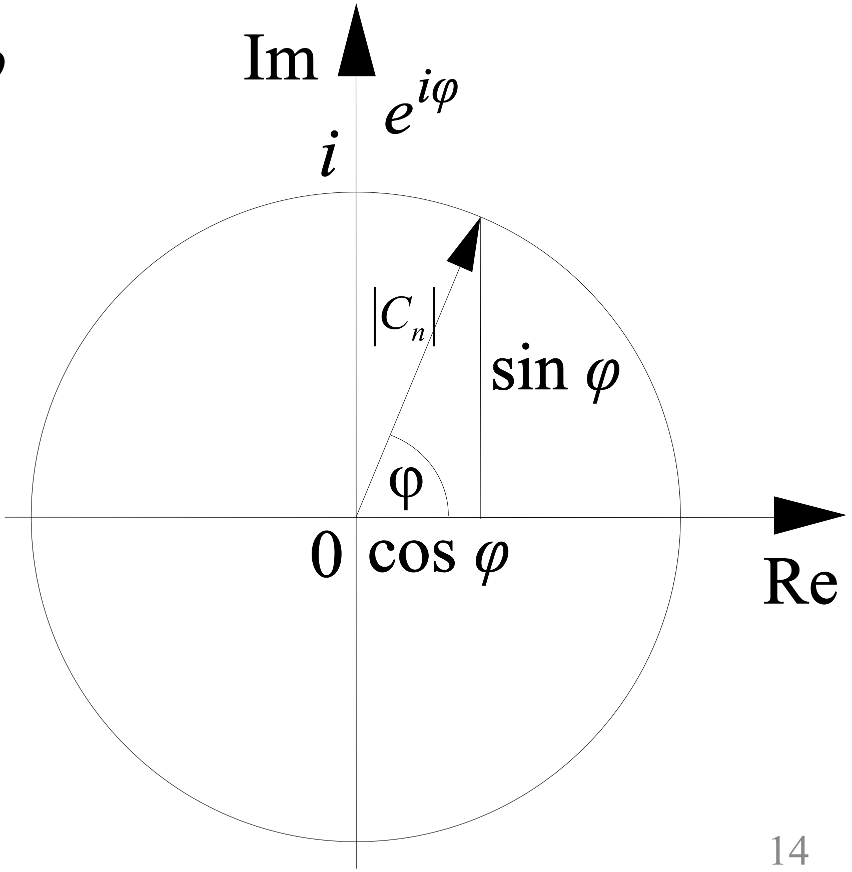
Polar form : $C_n = |C_n| e^{i\varphi}$

Magnitude : $|C_n|$

Phase : $\varphi$

$\mathrm{Re}(C_n) = |C_n| \cos\varphi$

$\mathrm{Im}(C_n) = |C_n| \sin\varphi$



EPFL Disal

# From Real to Complex Coefficients

*Real* Fourier coefficients:

$$f(t) = A_0 + \sum_{n=1}^{\infty}\left[A_n \cos\left(\frac{2\pi n t}{T}\right) + B_n \sin\left(\frac{2\pi n t}{T}\right)\right]; A_n, B_n \in \mathbb{R}$$

Using the Euler relationships:

$$e^{i\omega} = \cos\omega + i\sin\omega$$

$$e^{-i\omega} = \cos(-\omega) + i\sin(-\omega) = \cos\omega - i\sin\omega$$

we can express *sin* and *cos* using exponential functions

$$\Rightarrow \cos\omega = \frac{e^{i\omega} + e^{-i\omega}}{2}; \sin\omega = \frac{e^{i\omega} - e^{-i\omega}}{2i}$$

And substituting *sin* and *cos* we get ($\omega_0 = 2\pi/T$):

$$f(t) = A_0 + \sum_{n=1}^{\infty}\left[A_n\left(\frac{e^{in\omega_0 t} + e^{-in\omega_0 t}}{2}\right) + B_n\left(\frac{e^{in\omega_0 t} - e^{-in\omega_0 t}}{2i}\right)\right]$$

and then

$$f(t) = A_0 + \sum_{n=1}^{\infty}\left(\frac{A_n - iB_n}{2}\right)e^{in\omega_0 t} + \sum_{n=-\infty}^{-1}\left(\frac{A_n + iB_n}{2}\right)e^{in\omega_0 t}$$

We can now express our Fourier series with *complex* coefficients

$$f(t) = \sum_{n=-\infty}^{\infty} C_n e^{in\omega_0 t}; C_n \in \mathbb{C}$$

EPFL

Disal

# Complex Fourier Coefficients

**For 2π periodic function:**

Fourier series $$f(t)=\sum_{n=-\infty}^{\infty} C_n e^{in\omega_0 t}$$

Fourier coefficients $$C_n=\frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)e^{-in\omega_0 t}dt$$

Rem (s. 14): $C_n = |C_n|e^{i\varphi}$ Magnitude: $|C_n|$ Phase: φ

# Fourier Series for Arbitrary Period

$$f(t) = \sum_{n=0}^{\infty} A_n \cos\left(\frac{2\pi nt}{T}\right) + \sum_{n=1}^{\infty} B_n \sin\left(\frac{2\pi nt}{T}\right)$$

- Suppose f(t) periodic in T and integrable on this period
- The corresponding Fourier coefficients are:

$$A_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\cos(\frac{2\pi nt}{T})dt, \quad n \geq 0$$

$$B_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\sin(\frac{2\pi nt}{T})dt, \quad n \geq 1$$

Note: compare with s. 13 expressions with T=2π

# Complex Coefficient for Arbitrary Period

**Fourier series** $$f(t) = \sum_{n=-\infty}^{\infty} C_n e^{in\omega_0 t}$$

**Fourier coefficients** $$C_n = \frac{1}{T}\int_{-T/2}^{T/2} f(t) e^{-in\omega_0 t} dt$$

for T periodic function

Note: compare with s. 16 expressions with $T=2\pi$

Rem (s. 14): $C_n = |C_n| e^{i\varphi}$ Magnitude: $|C_n|$ Phase: $\varphi$

# Examples

## Matlab demo on Fourier series

**Associated to this book:**
J. H. McClellan, R. W. Schafer, M. A. Yoder
"DSP First: A Multimedia Approach", Prentice Hall, 1999.

**Available for download here:**

**https://dspfirst.gatech.edu/matlab/**

# Fourier Transform

# From Fourier Series to Transform

- f(t)
  - an aperiodic signal
  - view it as the limit of a periodic signal as $T \rightarrow \infty$
  - is a piecewise continuous, integrable (function space $L^1$) or even a square-integrable (function space $L^2$) function

$$\int_{-\infty}^{\infty} |f(x)|\, dx < \propto \qquad \int_{-\infty}^{\infty} |f(x)|^2\, dx < \infty$$

- For a periodic signal, the harmonic components are spaced $\omega_0 = 2\pi/T$ apart ($\omega_0$ = fundamental angular frequency)

- As $T \rightarrow \infty$, $\omega_0 \rightarrow 0$, and harmonic components are spaced closer and closer in frequency forming a continuous spectrum

# Fourier Transform

**Unitary, ordinary frequency notation**

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} f(t) \cdot e^{-i2\pi\xi t} dt$$

**Fourier Transform**

$$f(t) = \int_{-\infty}^{\infty} \hat{f}(\xi) \cdot e^{i2\pi t\xi} d\xi$$

**Inverse Fourier Transform**

**Notes:**

- f^ is often replaced with F
- the direct transform is also called “analysis equation”
- the inverse transform is also called “synthesis equation”

# Fourier Transform

**Non-unitary, angular frequency notation**

$$F(\omega) = \int_{-\infty}^{+\infty} f(t) e^{-i\omega t} dt$$

**Fourier Transform**

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} F(\omega) e^{i\omega t} d\omega$$

**Inverse Fourier Transform**

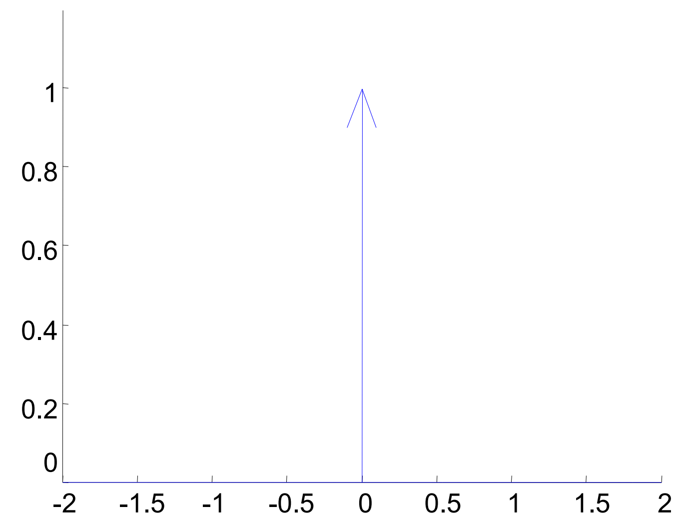
**Notes:**

- $\omega = 2\pi\xi \rightarrow$ obtained from unitary, ordinary frequency transform with $\xi = \omega/2\pi$
- F can be replaced with f^
- In electrical engineering *i* is substituted by *j* ("i" booked for current)
- Often, in order to emphasize the frequency response aspect, the imaginary aspect of the transform is emphasized: F(ω), F(iω), or F(jω) are all equivalent notations

# Dirac Delta Function

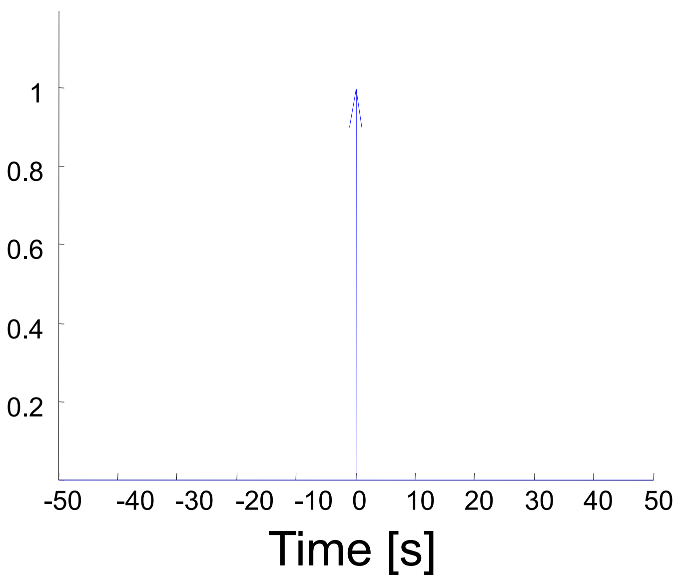
$$\delta(t) = \begin{cases} +\infty, & t = 0 \\ 0, & t \neq 0 \end{cases}$$

$$\int_{-\infty}^{+\infty} \delta(t)dt = 1$$




**Note:** The height of the arrow (or small circle) is usually used to specify the value of any multiplicative constant, which will give the area under the function; the other convention is to write the area next to the arrowhead. [Wikipedia]

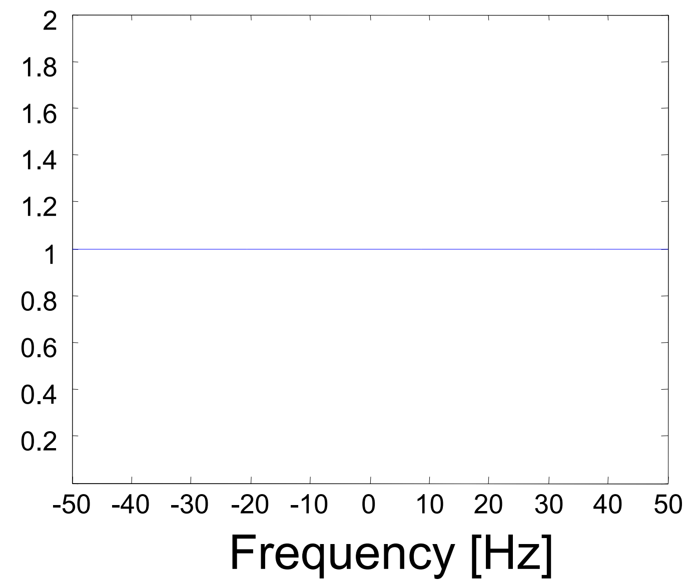
# Common Fourier Transform




$$f(t) = \delta(t)$$

FT
(analysis equation)

IFT
(synthesis equation)




$$\hat{f}(\xi) = 1$$

# FT of a Dirac Impulse

$$f(t) = \delta(t)$$

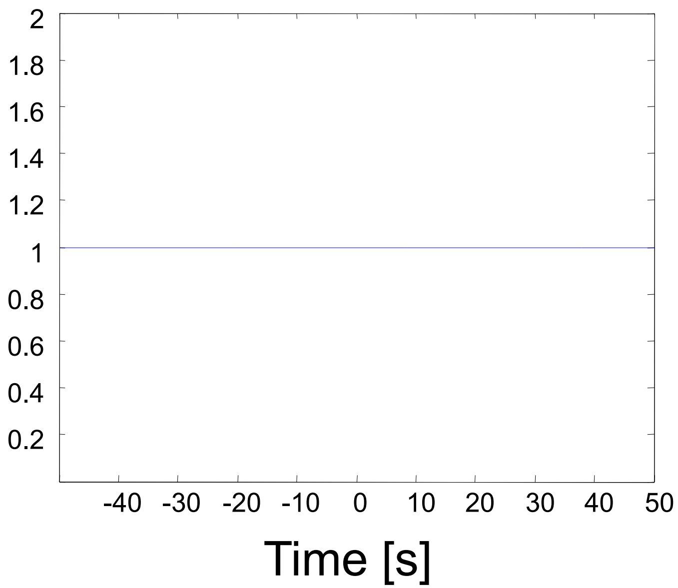
$$\hat{f}(\xi) = \int_{-\infty}^{+\infty} f(t)e^{-i2\pi\xi t}dt = \int_{-\infty}^{+\infty} \delta(t)e^{-i2\pi\xi t}dt = 1$$

Analysis equation

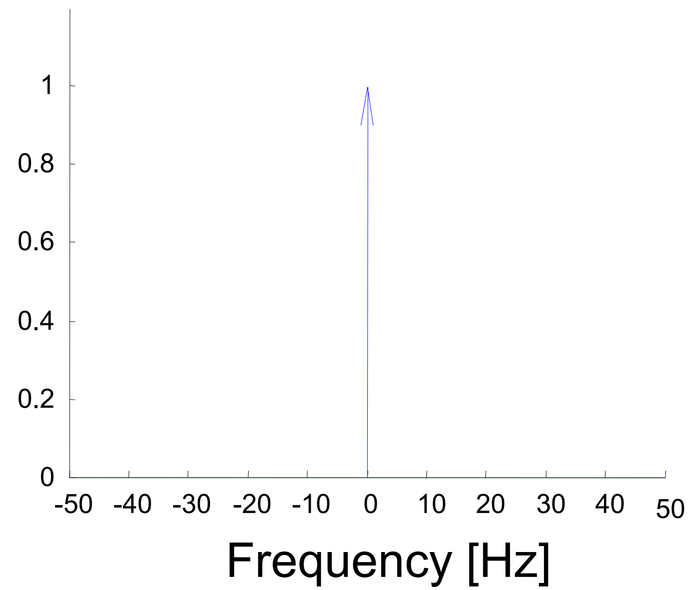
$$f(t) = \int_{-\infty}^{+\infty} \hat{f}(\xi)e^{i2\pi t\xi}d\xi = \int_{-\infty}^{+\infty} e^{i2\pi t\xi}d\xi = \delta(t)$$

Synthesis equation

# Common Fourier Transform

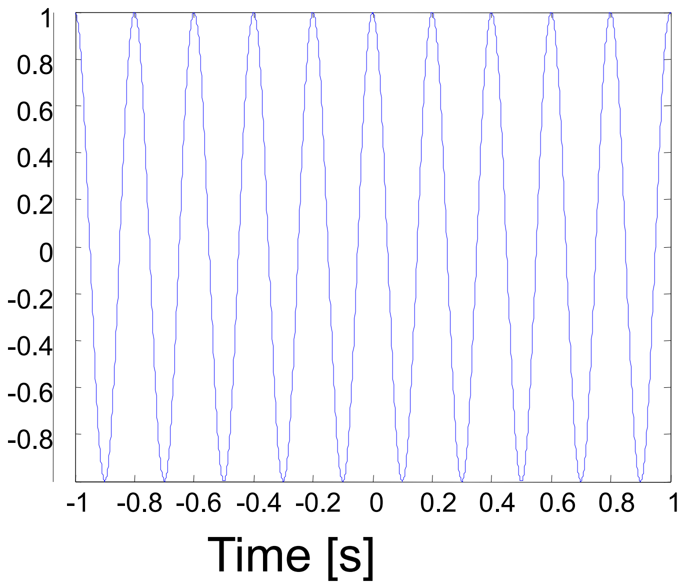


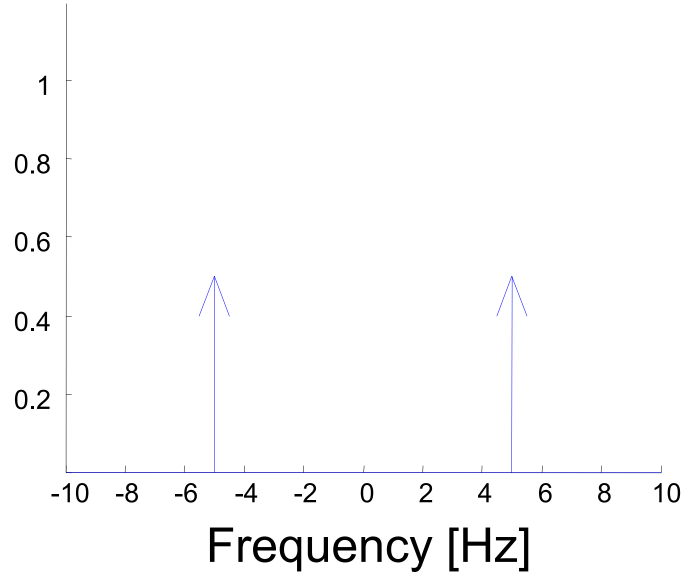

$$f(t) = 1$$




$$\hat{f}(\xi) = \delta(\xi)$$

# Common Fourier Transform

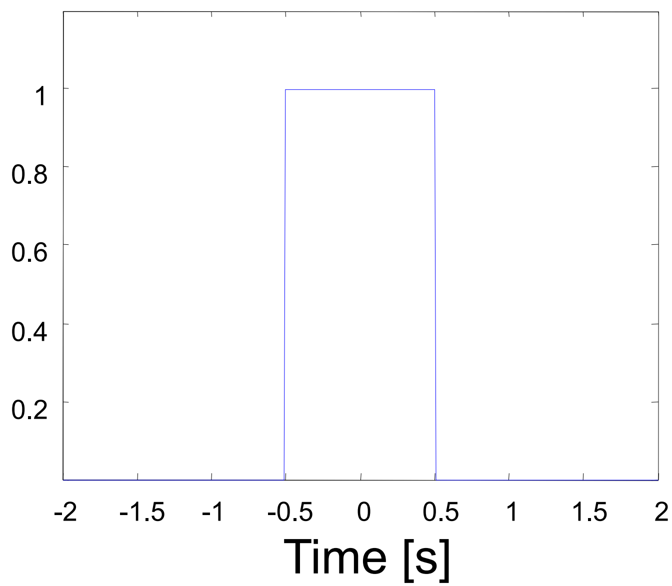


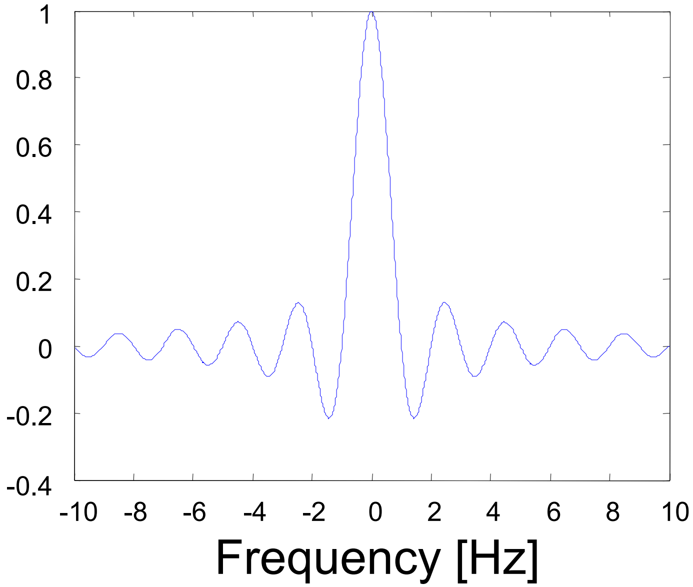




$$f(t) = \cos(2\pi a t)$$

$$\hat{f}(\xi) = \frac{\delta(\xi - a) + \delta(\xi + a)}{2}$$

# Common Fourier Transform




$$f(t) = rect(t)$$

$$\hat{f}(\xi) = \mathrm{sinc}(\xi)$$

# Properties

| | |
|---|---|
| **Linearity** | $h(t) = af(t) + bg(t) \Rightarrow \hat{h}(\xi) = a\hat{f}(\xi) + b\hat{g}(\xi)$ |
| **Translation** | $h(t) = f(t - t_0) \Rightarrow \hat{h}(\xi) = e^{-2\pi i t_0 \xi}\hat{f}(\xi)$ |
| **Modulation** | $h(t) = e^{2\pi i t \xi_0} f(t) \Rightarrow \hat{h}(\xi) = \hat{f}(\xi - \xi_0)$ |
| **Scaling** | $h(t) = f(at) \Rightarrow \hat{h}(\xi) = \frac{1}{\lvert a \rvert}\hat{f}(\frac{\xi}{a})$ |
| **Convolution** | $h(t) = (f * g)(t) \Rightarrow \hat{h}(\xi) = \hat{f}(\xi) \cdot \hat{g}(\xi)$ |

See s. 38 onwards

# Fourier Tables

# Fourier Transforms Table

Note x = t on s. 20/21

Note ν = ω on s. 21

Simple example s. 32

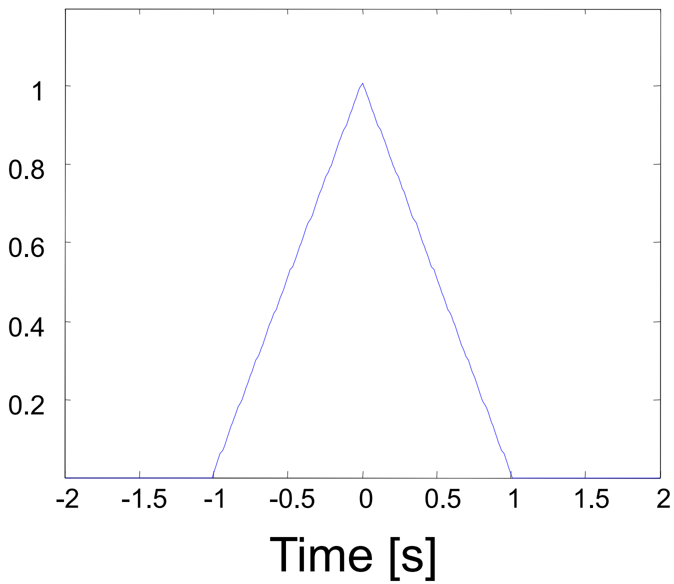
| | Function | Fourier transform unitary, ordinary frequency | Fourier transform unitary, angular frequency | Fourier transform non-unitary, angular frequency |
|---|---|---|---|---|
| | $f(x)$ | $\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i x \xi}\, dx$ | $\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\omega x}\, dx$ | $\hat{f}(\nu) = \int_{-\infty}^{\infty} f(x) e^{-i\nu x}\, dx$ |
| 201 | $\mathrm{rect}(ax)$ | $\frac{1}{\lvert a\rvert} \cdot \mathrm{sinc}\left(\frac{\xi}{a}\right)$ | $\frac{1}{\sqrt{2\pi a^2}} \cdot \mathrm{sinc}\left(\frac{\omega}{2\pi a}\right)$ | $\frac{1}{\lvert a\rvert} \cdot \mathrm{sinc}\left(\frac{\nu}{2\pi a}\right)$ |
| 202 | $\mathrm{sinc}(ax)$ | $\frac{1}{\lvert a\rvert} \cdot \mathrm{rect}\left(\frac{\xi}{a}\right)$ | $\frac{1}{\sqrt{2\pi a^2}} \cdot \mathrm{rect}\left(\frac{\omega}{2\pi a}\right)$ | $\frac{1}{\lvert a\rvert} \cdot \mathrm{rect}\left(\frac{\nu}{2\pi a}\right)$ |
| 203 | $\mathrm{sinc}^2(ax)$ | $\frac{1}{\lvert a\rvert} \cdot \mathrm{tri}\left(\frac{\xi}{a}\right)$ | $\frac{1}{\sqrt{2\pi a^2}} \cdot \mathrm{tri}\left(\frac{\omega}{2\pi a}\right)$ | $\frac{1}{\lvert a\rvert} \cdot \mathrm{tri}\left(\frac{\nu}{2\pi a}\right)$ |
| 204 | $\mathrm{tri}(ax)$ | $\frac{1}{\lvert a\rvert} \cdot \mathrm{sinc}^2\left(\frac{\xi}{a}\right)$ | $\frac{1}{\sqrt{2\pi a^2}} \cdot \mathrm{sinc}^2\left(\frac{\omega}{2\pi a}\right)$ | $\frac{1}{\lvert a\rvert} \cdot \mathrm{sinc}^2\left(\frac{\nu}{2\pi a}\right)$ |
| 205 | $e^{-ax}u(x)$ | $\frac{1}{a + 2\pi i \xi}$ | $\frac{1}{\sqrt{2\pi}(a + i\omega)}$ | $\frac{1}{a + i\nu}$ |
| 206 | $e^{-\alpha x^2}$ | $\sqrt{\frac{\pi}{\alpha}} \cdot e^{-\frac{(\pi\xi)^2}{\alpha}}$ | $\frac{1}{\sqrt{2\alpha}} \cdot e^{-\frac{\omega^2}{4\alpha}}$ | $\sqrt{\frac{\pi}{\alpha}} \cdot e^{-\frac{\nu^2}{4\alpha}}$ |
| 207 | $e^{-a\lvert x\rvert}$ | $\frac{2a}{a^2 + 4\pi^2\xi^2}$ | $\sqrt{\frac{2}{\pi}} \cdot \frac{a}{a^2 + \omega^2}$ | $\frac{2a}{a^2 + \nu^2}$ |

# Fourier Properties Table

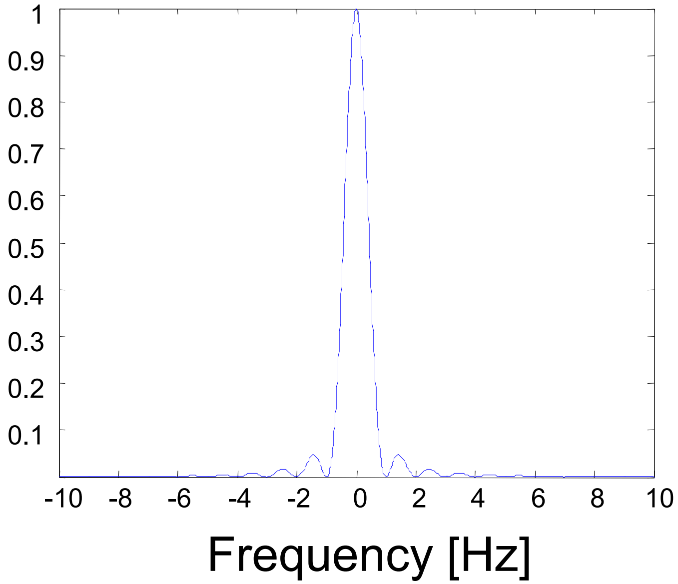
| | Function | Fourier transform unitary, ordinary frequency | Fourier transform unitary, angular frequency | Fourier transform non-unitary, angular frequency |
|---|---|---|---|---|
| | $f(x)$ | $\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i x \xi}\, dx$ | $\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\omega x}\, dx$ | $\hat{f}(\nu) = \int_{-\infty}^{\infty} f(x) e^{-i\nu x}\, dx$ |
| 101 | $a \cdot f(x) + b \cdot g(x)$ | $a \cdot \hat{f}(\xi) + b \cdot \hat{g}(\xi)$ | $a \cdot \hat{f}(\omega) + b \cdot \hat{g}(\omega)$ | $a \cdot \hat{f}(\nu) + b \cdot \hat{g}(\nu)$ |
| 102 | $f(x-a)$ | $e^{-2\pi i a \xi} \hat{f}(\xi)$ | $e^{-ia\omega} \hat{f}(\omega)$ | $e^{-ia\nu} \hat{f}(\nu)$ |
| 103 | $e^{2\pi i a x} f(x)$ | $\hat{f}(\xi - a)$ | $\hat{f}(\omega - 2\pi a)$ | $\hat{f}(\nu - 2\pi a)$ |
| 104 | $f(ax)$ | $\frac{1}{\|a\|} \hat{f}\left(\frac{\xi}{a}\right)$ | $\frac{1}{\|a\|} \hat{f}\left(\frac{\omega}{a}\right)$ | $\frac{1}{\|a\|} \hat{f}\left(\frac{\nu}{a}\right)$ |
| 105 | $\hat{f}(x)$ | $f(-\xi)$ | $f(-\omega)$ | $2\pi f(-\nu)$ |
| 106 | $\frac{d^n f(x)}{dx^n}$ | $(2\pi i \xi)^n \hat{f}(\xi)$ | $(i\omega)^n \hat{f}(\omega)$ | $(i\nu)^n \hat{f}(\nu)$ |
| 107 | $x^n f(x)$ | $\left(\frac{i}{2\pi}\right)^n \frac{d^n \hat{f}(\xi)}{d\xi^n}$ | $i^n \frac{d^n \hat{f}(\omega)}{d\omega^n}$ | $i^n \frac{d^n \hat{f}(\nu)}{d\nu^n}$ |
| 108 | $(f * g)(x)$ | $\hat{f}(\xi)\hat{g}(\xi)$ | $\sqrt{2\pi} \hat{f}(\omega)\hat{g}(\omega)$ | $\hat{f}(\nu)\hat{g}(\nu)$ |
| 109 | $f(x)g(x)$ | $(\hat{f} * \hat{g})(\xi)$ | $\frac{(\hat{f} * \hat{g})(\omega)}{\sqrt{2\pi}}$ | $\frac{1}{2\pi}(\hat{f} * \hat{g})(\nu)$ |

Combined example s. 33

# Simple Example

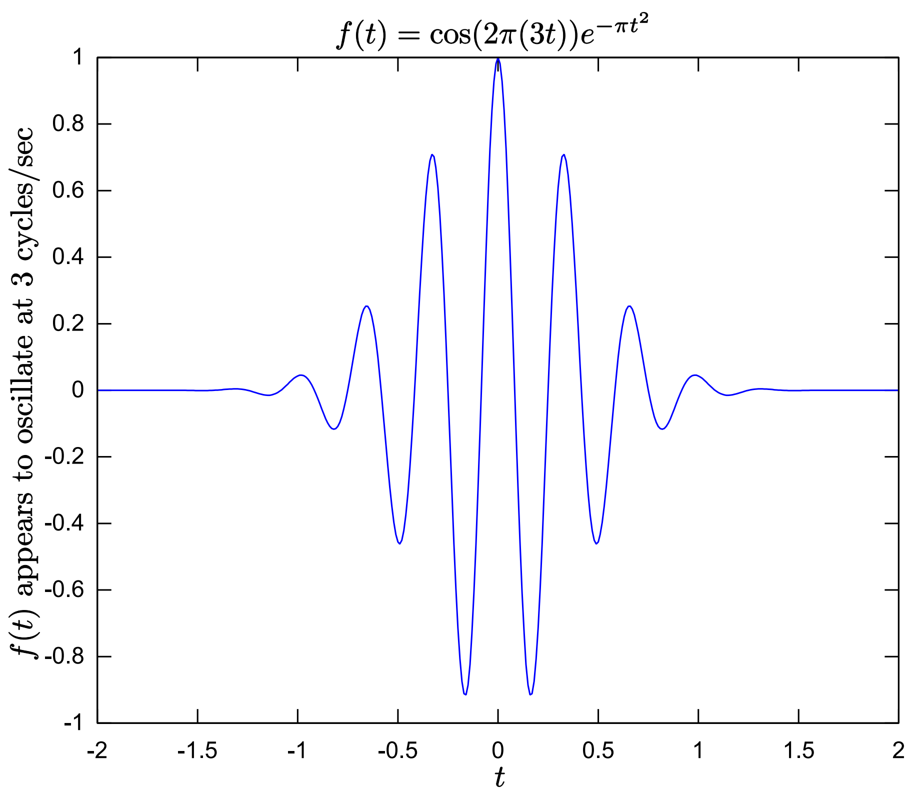


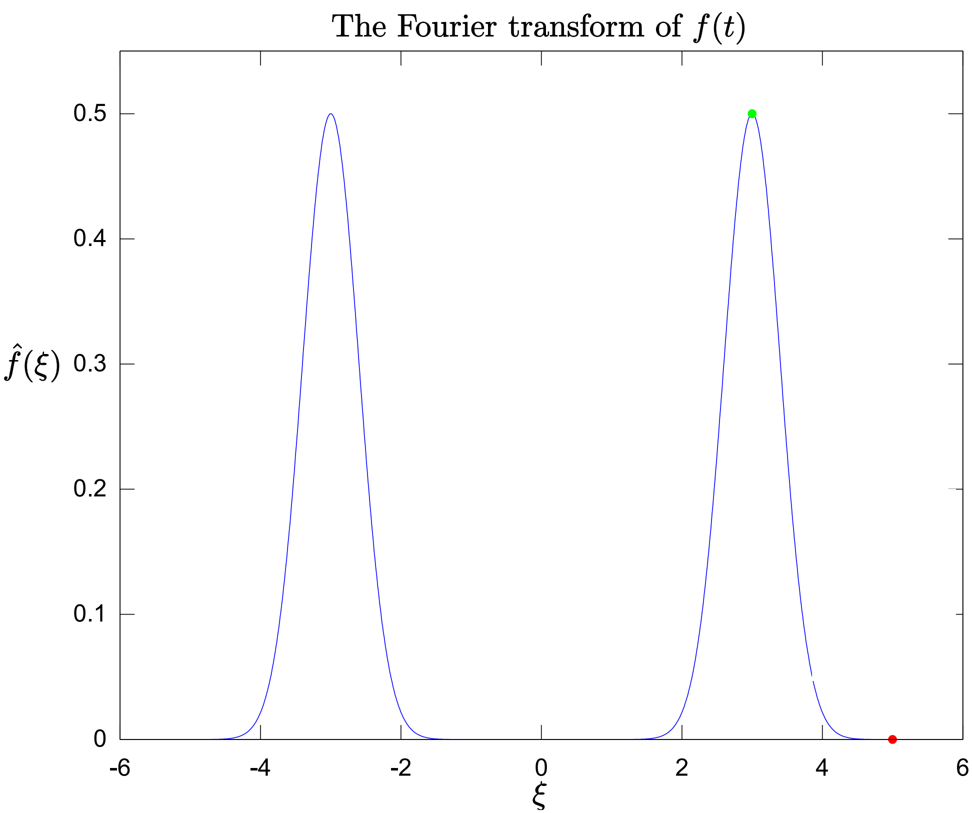

$$f(t) = trian(t)$$




$$\hat{f}(\xi) = \text{sinc}^2(\xi)$$

# Combined Example



# Discrete Fourier Transform

# Discrete Fourier Transform

$$\hat{f}(\xi) = F(\xi) = \int_{-\infty}^{\infty} f(t) \cdot e^{-2\pi i \xi t} dt$$

$$\downarrow$$

$$\mathrm{X[k]} = \sum_{n=0}^{N-1} x[n] \cdot e^{\frac{-2\pi i}{N} kn}, k = 0, \cdots, N-1$$

- Input and output sequences are both finite (the integral becomes a finite sum)
- Both time and frequency domain are discrete (note the "[]")
- Matlab, operating on a computer (i.e. a digital device) can only emulate continuity/infinity and therefore use this discrete version with an adjustable discretization level (in time, frequency, and amplitude) and finite bounds
- Also, since Matlab index start from 1, formulation is typically x[n+1], X[k+1]

# Discrete Fourier Transform

$$\mathrm{X[k]} = \sum_{n=0}^{N-1} \mathrm{x}[n] \cdot e^{\frac{-2\pi i}{N}kn}, k = 0, \cdots, N-1$$ **DFT**

$$\mathrm{x[n]} = \frac{1}{N} \sum_{k=0}^{N-1} \mathrm{X}[k] \cdot e^{\frac{2\pi i}{N}kn}, k = 0, \cdots, N-1$$ **Inverse DFT**

- Different from the Discrete-Time Fourier Transform (DTFT), see Week 4
- Efficient implementation, for instance in Matlab, is done using a Fast Fourier Transform (FFT) algorithm (e.g., Cooley and Tukey, 1965)
- Clear correspondence with the Fourier series (see slide 16)

# Differences among Fourier Series, Continuous and Discrete Transforms

**FT** $$\hat{f}(\xi) = F(\xi) = \int_{-\infty}^{\infty} f(t) \cdot e^{-2\pi i \xi t} dt$$

**DFT** $$\mathrm{X[k]} = \sum_{n=0}^{N-1} x[n] \cdot e^{\frac{-2\pi i}{N} kn}, k = 0, \cdots, N-1$$

**FS** $$f(t) = \sum_{n=-\infty}^{\infty} C_n e^{in\omega_0 t} \qquad C_n = \frac{1}{T} \int_{-T/2}^{T/2} f(t) e^{-in\omega_0 t} dt$$

# Convolution

# Convolution

$$(f * g)(t) = \int_{-\infty}^{\infty} f(\tau) \cdot g(t-\tau) d\tau$$

- For each value of $t$:
  1. Flip (reflect) $\boldsymbol{g}$
  2. Shift $\boldsymbol{g}$ by $\boldsymbol{t}$
  3. Multiply $\boldsymbol{f}$ and $\boldsymbol{g}$
  4. Integrate over $\boldsymbol{\tau}$

1) $g(\tau) \rightarrow g(-\tau)$

2) $g(-\tau) \rightarrow g(t-\tau)$

3) $f(\tau) \cdot g(t-\tau)$

4) $\int_{-\infty}^{\infty} f(\tau) \cdot g(t-\tau) d\tau$

- Note that the result does **not** depend on τ!

# Examples

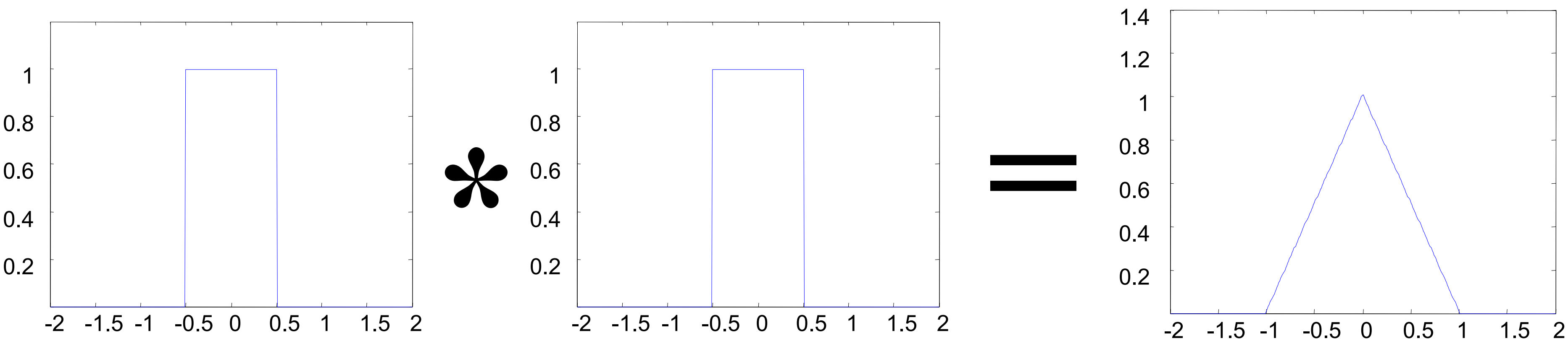
## Matlab demo on Continuous Convolution

**Associated to this book:**
J. H. McClellan, R. W. Schafer, M. A. Yoder
"DSP First: A Multimedia Approach", Prentice Hall, 1999.

**Available for download here:**

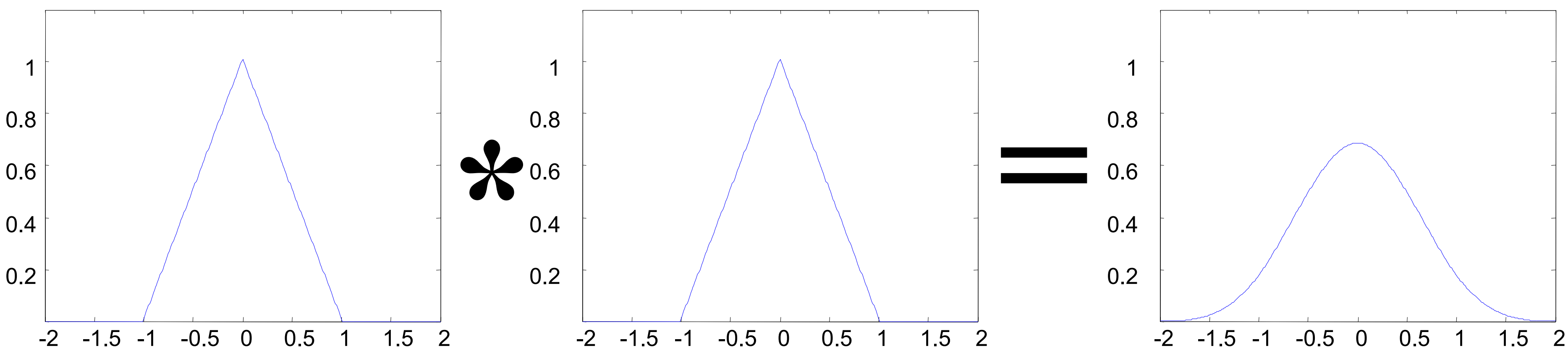
**https://dspfirst.gatech.edu/matlab/**

# Examples



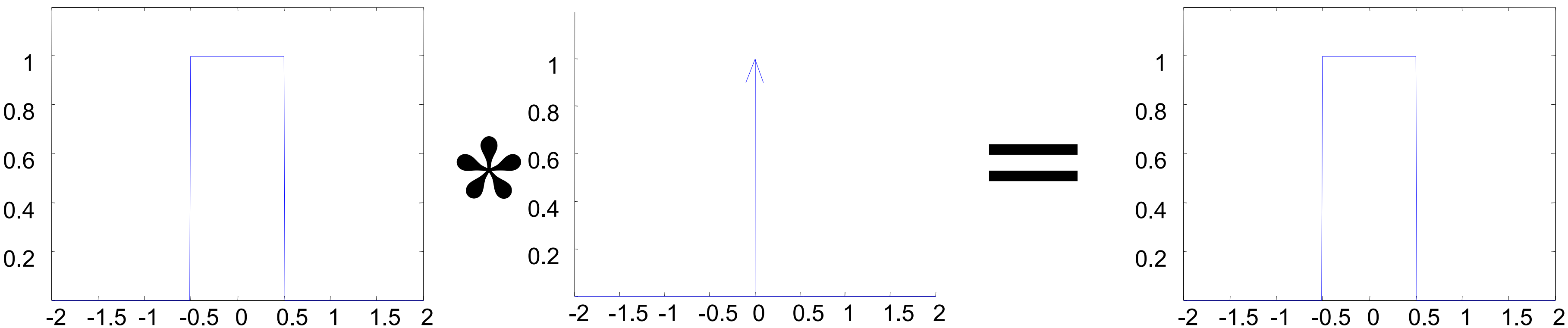
1
0.8
0.6
0.4
0.2
-2 -1.5 -1 -0.5 0 0.5 1 1.5 2
*
1
0.8
0.6
0.4
0.2
-2 -1.5 -1 -0.5 0 0.5 1 1.5 2
=
1.4
1.2
1
0.8
0.6
0.4
0.2
-2 -1.5 -1 -0.5 0 0.5 1 1.5 2

# Examples



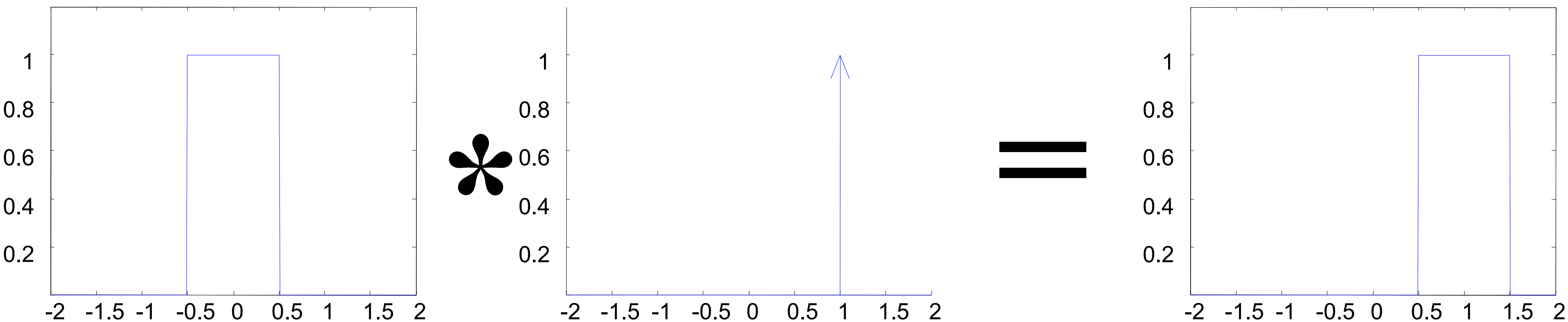
1
0.8
0.6
0.4
0.2
-2 -1.5 -1 -0.5 0 0.5 1 1.5 2
*
1
0.8
0.6
0.4
0.2
-2 -1.5 -1 -0.5 0 0.5 1 1.5 2
=
1
0.8
0.6
0.4
0.2
-2 -1.5 -1 -0.5 0 0.5 1 1.5 2

# Examples

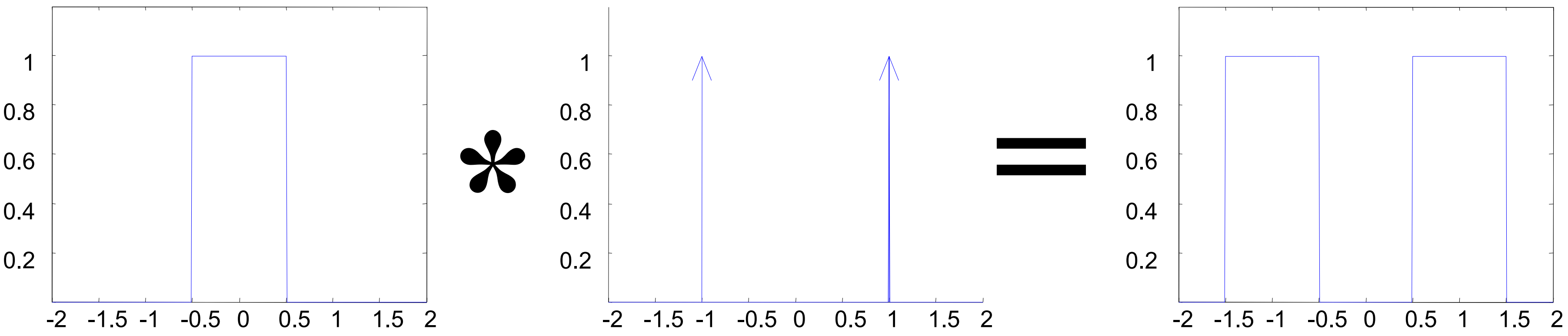


1
0.8
0.6
0.4
0.2
-2 -1.5 -1 -0.5 0 0.5 1 1.5 2
*
1
0.8
0.6
0.4
0.2
-2 -1.5 -1 -0.5 0 0.5 1 1.5 2
=
1
0.8
0.6
0.4
0.2
-2 -1.5 -1 -0.5 0 0.5 1 1.5 2

# Examples


1
0.8
0.6
0.4
0.2
-2 -1.5 -1 -0.5 0 0.5 1 1.5 2
*
1
0.8
0.6
0.4
0.2
-2 -1.5 -1 -0.5 0 0.5 1 1.5 2
=
1
0.8
0.6
0.4
0.2
-2 -1.5 -1 -0.5 0 0.5 1 1.5 2

# Examples



1
0.8
0.6
0.4
0.2
-2 -1.5 -1 -0.5 0 0.5 1 1.5 2
*
1
0.8
0.6
0.4
0.2
-2 -1.5 -1 -0.5 0 0.5 1 1.5 2
=
1
0.8
0.6
0.4
0.2
-2 -1.5 -1 -0.5 0 0.5 1 1.5 2

# Convolution in Time and Frequency Domains

**Time domain** **Frequency domain**

$$h(t) = (f * g)(t) \Leftrightarrow \hat{h}(\xi) = \hat{f}(\xi) \cdot \hat{g}(\xi)$$

$$h(t) = f(t) \cdot g(t) \Leftrightarrow \hat{h}(\xi) = (\hat{f} * \hat{g})(\xi)$$

Unitary, ordinary frequency

$$h(t) = (f * g)(t) \Leftrightarrow H(\omega) = F(\omega) \cdot G(\omega)$$

$$h(t) = f(t) \cdot g(t) \Leftrightarrow H(\omega) = \frac{1}{2\pi}(F * G)(\omega)$$

Non-unitary, angular frequency

# Convolution Properties

Commutativity

$$f * g = g * f$$

Associativity

$$f * (g * h) = (f * g) * h$$

Distributivity

$$f * (g + h) = (f * g) + (f * h)$$

Associativity with scalar multiplication

$$a(f * g) = (af) * g = f * (ag)$$

# Discrete Convolution

$$(f * g)(t) = \int_{-\infty}^{\infty} f(\tau) \cdot g(t-\tau) d\tau$$

$$\downarrow$$

$$(f * g)[n] = \sum_{m=-\infty}^{\infty} f[m] \cdot g[n-m]$$

**Notes:**

- Similar to the continuous version
- The integral becomes an infinite sum (note: in theory no bounds on the sum by definition as in the DFT)
- In practice, a computer (i.e. a digital device), running Matlab for instance, can only emulate continuity and therefore discrete time and quantized amplitude as well as finite bounds of the convolution window are used

# Examples

## Matlab demo on Discrete Convolution

**Associated to this book:**
J. H. McClellan, R. W. Schafer, M. A. Yoder
"DSP First: A Multimedia Approach", Prentice Hall, 1999.

**Available for download here:**

**https://dspfirst.gatech.edu/matlab/**

# Conclusion

# Take Home Messages

- Fourier decomposition: every periodic signal can be decomposed into a sum of sines and cosines
- Fourier coefficients are the weights in this sum
- Fourier Transform (FT) for aperiodic signals
- Implementation on computers of Fourier transform is discretized and bound: Discrete Fourier Transform (DFT)
- The Fast FT (FFT) is a fast algorithm for solving the DFT
- Properties of FT and tables help to solve analytically analysis and synthesis equations
- Multiplication in time domain means convolution in frequency domain and vice versa
- Continuous vs. discrete convolution (analogy: FT vs. DFT)

# Additional Literature – Week 2

## Related course material

- Information, Calcul, Communication (ICC)
- Analysis IV

## Books


- J. H. McClellan, R. W. Schafer, M. A. Yoder
  "DSP First: A Multimedia Approach", Prentice Hall, 1999.
- A. Oppenheim and A. S. Willsky with S. Nawab, "Signals and Systems", Prentice Hall, 1997.