Introduction to Signal Processing – Convolution, Sampling, Reconstruction
Motivation from Week 1 Lecture

Highlighted blocks are those mainly leveraging the content of this lecture.
Differences among Fourier Series, Continuous and Discrete Transforms

**FT**
\[ \hat{f}(\xi) = F(\xi) = \int_{-\infty}^{\infty} f(t) \cdot e^{-2\pi i \xi t} dt \]

**DFT**
\[ X[k] = \sum_{n=0}^{N-1} x[n] \cdot e^{-2\pi i kn/N}, \quad k = 0, \ldots, N - 1 \]

**FS**
\[ f(t) = \sum_{n=-\infty}^{\infty} C_n e^{i n \omega_0 t} \quad C_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) e^{-i n \omega_0 t} dt \]
Convolution
Convolution

\[(f * g)(t) = \int_{-\infty}^{\infty} f(\tau) \cdot g(t - \tau) d\tau\]

- For each value of \(t:\)
  1. Flip (reflect) \(g\)
  2. Shift \(g\) by \(t\)
  3. Multiply \(f\) and \(g\)
  4. Integrate over \(\tau\)

- Note that the result does not depend on \(\tau\)!
Examples

Matlab demo on Continuous Convolution

Associated to this book:
J. H. McClellan, R. W. Schafer, M. A. Yoder

Available for download here:

https://dspfirst.gatech.edu/matlab/
Examples
Examples
Examples
Examples
Examples

\[ \text{Graph 1} \ast \text{Graph 2} = \text{Result Graph} \]
Convolution in Time and Frequency Domains

Time domain

\[ h(t) = (f \ast g)(t) \iff \hat{h}(\xi) = \hat{f}(\xi) \cdot \hat{g}(\xi) \]

\[ h(t) = f(t) \cdot g(t) \iff \hat{h}(\xi) = (\hat{f} \ast \hat{g})(\xi) \]

Frequency domain

\[ h(t) = (f \ast g)(t) \iff H(\omega) = F(\omega) \cdot G(\omega) \]

\[ h(t) = f(t) \cdot g(t) \iff H(\omega) = \frac{1}{2\pi} (F \ast G)(\omega) \]
Convolution Properties

Commutativity
\[ f * g = g * f \]

Associativity
\[ f * (g * h) = (f * g) * h \]

Distributivity
\[ f * (g + h) = (f * g) + (f * h) \]

Associativity with scalar multiplication
\[ a(f * g) = (af) * g = f * (ag) \]
Discrete Convolution

\[
(f * g)(t) = \int_{-\infty}^{\infty} f(\tau) \cdot g(t - \tau) d\tau
\]

\[
(f * g)[n] = \sum_{m=-\infty}^{\infty} f[m] \cdot g[n - m]
\]

**Notes:**
- Similar to the continuous version
- The integral becomes an infinite sum (note: in theory no bounds on the sum by definition as in the DFT)
- In practice, a computer (i.e. a digital device), running Matlab for instance, can only emulate continuity and therefore discrete time and quantized amplitude as well as finite bounds of the convolution window are used
Examples

Matlab demo on Discrete Convolution

Associated to this book:
J. H. McClellan, R. W. Schafer, M. A. Yoder

Available for download here:

https://dspfirst.gatech.edu/matlab/
Sampling
Analog-Digital Converter (ADC)

- Transforms continuous analog signal into series of values
- Two key elements
  - **Sampling** (in time)
  - **Quantization** (of values)

\[ y[n] = 0 \ 0 \ -2 \ -4 \ -2 \ 0 \ 4 \ 8 \ 10 \ 10 \]
Periodic Train of Dirac Impulses

\[ x(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT) \]
Sampling in Time Domain

\[ x(t) \]

\[ p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT) \]

Periodic train of Dirac pulses

\[ x_p(t) = x(t)p(t) = \sum_{n=-\infty}^{\infty} x(t)\delta(t - nT) = \sum_{n=-\infty}^{\infty} x(nT)\delta(t - nT) \]

T = sampling period

\( f_s = 1/T = \text{sampling frequency} \)
Sampling in the Frequency Domain

• It would be nice to understand what does it mean sampling in the frequency domain so that we can leverage this representation for further reasoning (e.g., filter design)
• Multiplication in time domain means convolution in frequency domain but …
• How does look like a periodic train of Dirac impulses in the frequency domain?
From W4 (s.41): Fourier Series with Complex Fourier Coefficients

**Fourier series**

\[ f(t) = \sum_{n=-\infty}^{\infty} C_n e^{in\omega_0 t} \]

**Fourier coefficients**

\[ C_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t)e^{-in\omega_0 t} dt \]

**Rem:**

\[ C_n = |C_n| e^{i\phi} \]

Magnitude: \( |C_n| \)

Phase: \( \phi \)
From W4 (s.46): Fourier Transform

Non-unitary, angular frequency notation

\[
F(\omega) = \int_{-\infty}^{+\infty} f(t) e^{-i\omega t} \, dt
\]

Fourier Transform

\[
f(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} F(\omega) e^{i\omega t} \, d\omega
\]

Inverse Fourier Transform

Notes:

- \( \omega = 2\pi \xi \rightarrow \) obtained from unitary, ordinary frequency transform with \( \xi = \omega / 2\pi \)
- F can be replaced with \( f^\wedge \)
- In electrical engineering \( i \) is substituted by \( j \) ("i" booked for current)
- Often, in order to emphasize the frequency response aspect, the imaginary aspect of the transform is emphasized: \( F(\omega), F(i\omega), \) or \( F(j\omega) \) are all equivalent notations
From W4 (s. 47): FT of Dirac Impulse

\[ f(t) = \delta(t) \]

\[ \hat{f}(\xi) = 1 \]
FT for Periodic Signals

• Although it generalizes to aperiodic signals, the FT can be also applied to periodic signals.

• We can derive the FT of a periodic signal from its Fourier series (which have been developed for periodic signals, see W4).

• The FT of a periodic Dirac train of impulses consists of a periodic train of impulses in the frequency domain as well, with the area of impulses proportional to the Fourier series coefficients.

From [Oppenheim et al., 1997]
FT for Periodic Signals

(assume: period \( T \), \( \omega_0 = 2\pi/T \))

\[
X(\omega) = 2\pi\delta(\omega - \omega_0)
\]

\[
x(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} 2\pi\delta(\omega - \omega_0) e^{i\omega t} \, d\omega
\]

\[
x(t) = e^{i\omega_0 t}
\]

Linear combination of impulses equally spaced in frequency:

\[
X(\omega) = \sum_{n=-\infty}^{+\infty} 2\pi C_n \delta(\omega - n\omega_0)
\]

\[
x(t) = \sum_{n=-\infty}^{+\infty} C_n e^{in\omega_0 t}
\]

Compare with W4, s. 41, Fourier series
FT for Periodic Impulse Train

(assume: period $T$, $\omega_0 = 2\pi / T$)

Now consider the periodic impulse train of before:

$$x(t) = \sum_{n=-\infty}^{+\infty} \delta(t - nT)$$

Calculate the coefficient of its Fourier series:

$$C_n = \frac{1}{T} \int_{-T/2}^{+T/2} \delta(t)e^{-in\omega_0 t} \, dt = \frac{1}{T}$$

This means that each Fourier coefficient of the periodic impulse train has the same value; insert $C_n$ in previous expression (s. 25):

$$X(\omega) = \sum_{n=-\infty}^{+\infty} 2\pi C_n \delta(\omega - n\omega_0)$$

$$X(\omega) = \sum_{n=-\infty}^{+\infty} \frac{2\pi}{T} \delta(\omega - n\omega_0) = \frac{2\pi}{T} \sum_{n=-\infty}^{+\infty} \delta(\omega - n\omega_0)$$
Train of Dirac Impulses

\[ x(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT) \]

\[ X(\omega) = \sum_{k=-\infty}^{\infty} \frac{2\pi}{T} \delta(\omega - k\omega_s) \]

\[ \omega_s = \frac{2\pi}{T} \]

Sampling angular frequency
Sampling in Frequency Domain

Time domain

\[ x_p(t) = x(t)p(t) \]

Frequency domain

\[ X_p(\omega) = \frac{1}{2\pi} X(\omega) * P(\omega) \]

\[ P(\omega) = \frac{2\pi}{T} \sum_{k=-\infty}^{\infty} \delta(\omega - k\omega_s) \]

\[ X_p(\omega) = \frac{1}{T} \sum_{k=-\infty}^{\infty} X(\omega) * \delta(\omega - k\omega_s) \]

\[ = \frac{1}{T} \sum_{k=-\infty}^{\infty} X(\omega - k\omega_s) \]

Note: see also ss. 9-11 as examples for this operation
Sampling a Band-Limited Signal

$X(\omega) \rightarrow$ spectrum of signal $x(t)$ with highest frequency $< \omega_m$

$$P(\omega) = \frac{2\pi}{T} \sum_{k=-\infty}^{\infty} \delta(\omega - k\omega_s)$$

Angular sampling frequency $\omega_s > 2\omega_m$

$$X_p(\omega) = \frac{1}{2\pi} X(\omega) * P(\omega)$$

$$= \frac{1}{T} \sum_{k=-\infty}^{\infty} X(\omega - k\omega_s)$$
Sampling a Band-Limited Signal

\[ X(\xi) \rightarrow \text{spectrum of signal } x(t) \]
with highest frequency < 2 kHz

Sampling frequency: 5 kHz > 2 \times 2 \text{ kHz}
Sampling a Band-Limited Signal

$X(\xi) \rightarrow$ spectrum of signal $x(t)$ with highest frequency $< 3 \text{ kHz}$

Sampling frequency: $5 \text{ kHz} < 2 \times 3 \text{ kHz}$
\[ f(t) = \sin(2\pi t) + 0.4 \sin(2\pi \cdot 2t) + 0.2 \sin(2\pi \cdot 5t) \]
Too Few Samples (1Hz)

→ Data is lost
Too Many Samples (100 Hz)

→ Redundant data
→ Increase of data size
Minimal Possible Sampling (> 10 Hz)
Nyquist–Shannon Theorem

- If a function $x(t)$ contains no frequencies higher than $B$ Hz, it is completely determined by giving its coordinates at a series of points spaced $1/(2B)$ seconds apart.
- Sampling frequency must be at least two times greater than the maximal signal frequency.
Sampling in Practice

• Sampling frequency two times greater than maximal frequency is the limit
• Example: audio CD, sampling at 44.1 kHz since maximal hearable frequency: 20 kHz
• If possible, try to use a sampling frequency 10 times greater than the maximal frequency (help all sorts of filtering and reconstruction processes)
Signal Reconstruction
\[ p(t) = \sum_{n=-\infty}^{+\infty} \delta(t-nT) \]

spectrum of original signal

\[ x(t) \xrightarrow{X} x_p(t) \xrightarrow{H(\omega)} x_r(t) \]

spectrum of sampled signal

sampling angular frequency \( \omega_s > 2 \omega_m \)

filtering

filter cut-off angular frequency \( \omega_m < \omega_c < (\omega_s - \omega_m) \)

\[ X_r(\omega) = X_p(\omega)H(\omega) \]

spectrum of reconstructed signal

If there is no overlap between the shifted spectra the signal \( x_r(t) \) can be perfectly reconstructed from \( x(t) \)
$p(t) = \sum_{n=-\infty}^{+\infty} \delta(t - nT)$

- **spectrum of original signal**
- **spectrum of sampled signal**
  - sampling frequency $f_s < 2 f_m$
- **filtering**
  - filter cut-off frequency $f_c$
  - $f_m < f_c < (f_s - f_m)$

$x(t) \xrightarrow{X} x_p(t) \xrightarrow{H(\omega)} x_r(t)$

If there is overlap between the shifted spectra the signal $x_r(t)$ cannot be perfectly reconstructed from $x(t)$
Time Domain Interpretation of Signal Reconstruction

\[ p(t) = \sum_{n=-\infty}^{+\infty} \delta(t - nT) \]

- Multiplication with a Low-Pass Filter (LPF) in the frequency domain
- The LPF interpolates the samples assuming \( x(t) \) contains no energy at frequencies \( > \omega_c \) (\( \omega_c = \) cutoff angular frequency)

\[ x_r(t) = x_p(t) * h(t) \]

\[ = \left( \sum_{n=-\infty}^{\infty} x(nT) \delta(t - nT) \right) * h(t) \]

\[ = \sum_{n=-\infty}^{\infty} x(nT) h(t - nT) \]

\[ = \sum_{n=-\infty}^{\infty} x(nT) \frac{T \sin(\omega_c (t - nT))}{\pi(t - nT)} \]

Note: see also ss. 9-11 as examples for this operation
Signal Reconstruction in Practice

1. **Whittaker-Shannon interpolation** (band-limited interpolation):

   - Signal has to be band limited
     (i.e. Fourier transform for frequencies greater than \( B \) equal 0)
   - The sampling rate must exceed twice the bandwidth, \( 2B \), i.e. \( f_s > 2B \)

   Assume: \( \omega_c = \frac{\omega_s}{2} = \frac{2\pi}{2T} = \frac{\pi}{T} \) in s. 41, \( \text{sinc}(x) = \frac{\sin(\pi x)}{\pi x} \) Normalized sinc function

   \[
x_r(t) = \sum_{n=-\infty}^{\infty} x(nT) \cdot \text{sinc} \left( \frac{t - nT}{T} \right)
   \]

   \[
x_r(t) = \left( \sum_{n=-\infty}^{\infty} x(nT) \cdot \delta(t - nT) \right) \ast \text{sinc} \left( \frac{t}{T} \right) \quad \text{(Alternative equivalent formulation)}
   \]
Graphic Illustration of Time-Domain Interpolation

Original CT signal

After sampling

After passing the LPF (Low Pass Filter)

From Prof. A. S. Willsky, Signals and Systems course
Signal Reconstruction in Practice

2. Zero-order hold

3. First-order hold (linear interpolation)

From Prof. A. S. Willsky, Signals and Systems course
Aliasing
No Problems in Reconstruction
Reconstruction Problems

Overlapping

Alias

\[ X(f) \]

\[ X_A(f) \]
Harmonics

The diagram above illustrates the harmonic frequencies of a signal over time. The x-axis represents time in seconds, and the y-axis represents the amplitude of the signal. The diagram shows the following harmonics:

- **1Hz**: Blue line
- **2Hz**: Red line
- **3Hz**: Black line
- **4Hz**: Green line

Each harmonic is plotted at different frequencies, with the amplitude varying across the time frame from -0.5 to 0.5 seconds.
Harmonics

Fundamental Frequency

0 1/2 1

1/3

1/4

1/5

1/6

1/7
La-Tone (440 Hz) sampled at 44.1 kHz (CD standard)

Rem: s. 30, W04, max audible signals by humans 20 kHz
La-Tone (440 Hz) sampled at 2 kHz

Anti-alias filter: desired cut-off frequency 1 kHz

Reduced sampling frequency: 2 kHz

Actual high-order digital filter (see next week)
La-Tone (440 Hz) sampled at 2 kHz without filtering

Original signal with aliasing effect (A)

Repeated & shifted spectrum through sampling
La-Tone (440 Hz) sampled at 2 kHz filtered at 1 kHz

Original signal without aliases through anti-alias filtering

Repeated & shifted spectrum through sampling but without aliasing (anti-alias filter); will be eliminated through low-pass filtering at signal reconstruction stage
Aliasing Audio Examples

Original sound  Aliases 4 kHz  Correct sampling 4 kHz

Note: for instance cymbals (8-16 kHz) fully cut off, no matter what
Moiré Pattern
Aliasing Video Examples

https://www.youtube.com/watch?v=jHS9JGkEOmA
https://www.youtube.com/watch?v=R-IVw8OKjvQ
https://www.youtube.com/watch?v=jHS9JGkEOmA
Conclusion
Take Home Messages

• Multiplication in time domain means convolution in frequency domain and vice versa
• Continuous vs. discrete convolution (analogy: FT vs. DFT)
• Sampling: multiplication of the signal with a periodic train of Dirac pulses and therefore repeated shifted spectrum in the frequency domain
• Aliasing: higher frequency “folded back” on original spectrum -> prevent proper signal reconstruction
• When you sample a signal …
  – Make sure you know what the maximum frequency \( f_{\text{max}} \) is or enforce it through an anti-alias low-pass filter
  – Make sure you sample at \( f_s > 2 f_{\text{max}} \) (Nyquist-Shannon theorem)
Additional Literature – Week 5

• **Books:**

  • J. H. McClellan, R. W. Schafer, M. A. Yoder