Signals, Instruments, and Systems – W4

An Introduction to Signal Processing – Signals, Series, Transforms
Signal – Definition

“A signal is any time-varying or spatial-varying quantity”
Signal – Definition

“A signal is any time-varying or spatial-varying quantity”

\[ T = f(t) \]
Signal – Definition

“A signal is any \textbf{time}-varying or \textbf{spatial}-varying quantity”

\[ h = f(x) \]
Signal – Definition

“A signal is any **time**-varying or **spatial**-varying quantity”
Signal – Definition

“A signal is any **time**-varying or **spatial**-varying quantity”

\[ h = f(x, y) \]
Signal – Definition

“A signal is any time-varying or spatial-varying quantity”

\[ I = f[x, y] \]
Continuous versus Discrete
Continuous

- Represents values from the set of real numbers ($\mathbb{R}$).

Discrete

- Represents values from the set of natural numbers ($\mathbb{N}$).

Example:
- Continuous: $x \in \mathbb{R}$, $x = 2.478956983748...$
- Discrete: $x \in \mathbb{N}$, $x = 1, 2, 3, 4, 5, 6$
Analog versus Digital
Analog versus Digital

• An analog signal …
  – is **continuous** in time
  – has **continuous** values

• A digital signal …
  – is **discrete** in time
  – has **discrete** values
Analog – Digital

• Analog
  – Continuous
  – Real world

• Digital
  – Discrete
  – Digital world
# Sensors Classification

Continuous – Discrete

<table>
<thead>
<tr>
<th></th>
<th>Continuous</th>
<th>Discrete</th>
</tr>
</thead>
<tbody>
<tr>
<td>Time</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Space</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Amplitude</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Continuous: Time and Amplitude

Barometer

Thermometer
Continuous: Time and Amplitude

Barograph

Gramophone record

Thermograph
Continuous: Space and Amplitude
Discrete: Time

Camera

Reel camera
Discrete: Time and Amplitude

Weather station

Digital Watch
Discrete: Time, Space and Amplitude

Digital camera

Digital video camera
Analog-Digital-Analog Conversion

Analog to Digital Conversion (ADC)

Digital to Analog Conversion (DAC)

\[ y = x(t) \quad y = x[n] \]
Analog-Digital Converter (ADC)

- Transforms continuous analog signal into series of values
- Two key elements
  - Sampling (in time)
  - Quantization (of values)
- Two types of converters
  - Linear ADCs
  - Non-linear ADCs

\[ y[n] = 0 \ 0 \ -2 \ -4 \ -2 \ 0 \ 4 \ 8 \ 10 \ 10 \]
Examples of ADC Applications

- Input line of a soundcard
- Sensor of digital camera
- Mobile phone
- Computer mouse
- ...

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An Example from your Course Tools

- From datasheet dsPIC on e-puck: ADC 16 channels, 12 bits, 200 ksp/s total capacity
- \(2^{12}\) -1 discrete levels of resolution per channel
- Max 200k Hz sampling frequency on 1 channel -> e.g., 10 channels: 20k Hz
Digital-Analog Conversion
Digital-Analog Converter (DAC)

- Transforms a series of values into a continuous signal
- DAC outputs piecewise constant signal
- Additional filter stage to smooth signal (often carried out by the properties of the transducer/actuator itself)

\[ y[n] = 0 \quad 0 \quad -2 \quad -4 \quad -2 \quad 0 \quad 4 \quad 8 \quad 10 \quad 10 \]
Examples of DAC Applications

- Mobile phone
- MP3 player
- Graphics card (with older monitors)
- Monitors (newer systems)
Typical Waves and Signals
“A wave is a disturbance that propagates through space and time, usually with transference of energy.”

Wave function:

\[ f(x,t) = \gamma \sin(kx - \omega t) \]

- \( k = \frac{2\pi}{\lambda} \) “wave number”
- \( \omega = 2\pi f \) “angular frequency”
- \( \lambda = \frac{v}{f} \) “wave length”

\( \gamma = \text{wavelength} \quad v = \text{speed in x direction} \quad y = \text{amplitude} \)
Electromagnetic Spectrum

Increasing Wavelength ($\lambda$) $\rightarrow$

Increasing Frequency ($\nu$) $\leftarrow$

$\nu$ (Hz)

$\lambda$ (m)

$10^{24}$ $10^{22}$ $10^{20}$ $10^{18}$ $10^{16}$ $10^{14}$ $10^{12}$ $10^{10}$ $10^{8}$ $10^{6}$ $10^{4}$ $10^{2}$ $10^{0}$

$10^{-16}$ $10^{-14}$ $10^{-12}$ $10^{-10}$ $10^{-8}$ $10^{-6}$ $10^{-4}$ $10^{-2}$ $10^{0}$ $10^{2}$ $10^{4}$ $10^{6}$ $10^{8}$

$\gamma$ rays X rays UV IR Microwave FM Radio waves AM Long radio waves

Visible spectrum

Increasing Wavelength ($\lambda$) in nm $\rightarrow$
Common Frequencies

- Car motor: ~50 Hz (3000 rpm)
- Power lines: 50 Hz
- Ear: 20 Hz – 20 kHz
- Ultrasound: 20 kHz – 200 MHz
- Medical Sonograph: ~2-18 MHz
- Watch quartz: 32 kHz
- CPU: 2-3 GHz
- GSM: 0.9/1.8 GHz
Standard Waveforms

- Sine
- Square
- Triangle
- Sawtooth
Sinusoid (Sine Wave) in Time

\[ y(t) = A \cdot \sin(\omega t + \theta) \]

\( A \) : amplitude
\( \omega \) : angular frequency
\( \theta \) : phase
Dirac Delta Function

\[ \delta(t) = \begin{cases} + \infty, & t = 0 \\ 0, & t \neq 0 \end{cases} \]

\[ \int_{-\infty}^{+\infty} \delta(t) \, dt = 1 \]

**Note:** The height of the arrow (or small circle) is usually used to specify the value of any multiplicative constant, which will give the area under the function; the other convention is to write the area next to the arrowhead. [Wikipedia]
Fourier Series
Fourier Series

• Joseph Fourier (1768-1830) proposed that any periodic function can be decomposed into a sum of simple oscillating functions, namely sines and cosines.
“Any periodic function …”

from: Robert W. Stewart, Daniel García-Alís, “Concise DSP Tutorial”
"…can be decomposed into a sum of sines and cosines"

from: Robert W. Stewart, Daniel García-Alís, “Concise DSP Tutorial”
Fourier Series

\[ f(t) = \sum_{n=0}^{\infty} A_n \cos\left(\frac{2\pi nt}{T}\right) + \sum_{n=1}^{\infty} B_n \sin\left(\frac{2\pi nt}{T}\right) \]

• Suppose a \(2\pi\) periodic function \(f(t)\) integrable on \([-\pi, \pi]\)

• The Fourier coefficients of \(x\) are:

\[
A_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos(nt) dt, \quad n \geq 0
\]

\[
B_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \sin(nt) dt, \quad n \geq 1
\]
Complex Numbers Review

Rectangular form: $C_n = \text{Re}(C_n) + i \text{Im}(C_n)$

Euler's formula: $e^{i\varphi} = \cos \varphi + i \sin \varphi$

Polar form: $C_n = |C_n|e^{i\varphi}$

Magnitude: $|C_n|$

Phase: $\varphi$

$\text{Re}(C_n) = |C_n|\cos \varphi$

$\text{Im}(C_n) = |C_n|\sin \varphi$
From Real to Complex Coefficients

Real Fourier coefficients:

\[ f(t) = A_0 + \sum_{n=1}^{\infty} \left[ A_n \cos\left(\frac{2\pi nt}{T}\right) + B_n \sin\left(\frac{2\pi nt}{T}\right) \right]; A_n, B_n \in \mathbb{R} \]

Using the Euler relationships:

\[ e^{i\omega} = \cos \omega + i \sin \omega \]
\[ e^{-i\omega} = \cos(-\omega) + i \sin(-\omega) = \cos \omega - i \sin \omega \]

we can express \( \sin \) and \( \cos \) using exponential functions

\[ \Rightarrow \cos \omega = \frac{e^{i\omega} + e^{-i\omega}}{2}; \sin \omega = \frac{e^{i\omega} - e^{-i\omega}}{2i} \]

And substituting \( \sin \) and \( \cos \) we get \((\omega_0 = 2\pi/T)\):

\[ f(t) = A_0 + \sum_{n=1}^{\infty} \left[ A_n \left(\frac{e^{in\omega_0 t} + e^{-in\omega_0 t}}{2}\right) + B_n \left(\frac{e^{in\omega_0 t} - e^{-in\omega_0 t}}{2i}\right) \right] \]

and then

\[ f(t) = A_0 + \sum_{n=1}^{\infty} \left(\frac{A_n - iB_n}{2}\right)e^{in\omega_0 t} + \sum_{n=-\infty}^{-1} \left(\frac{A_n + iB_n}{2}\right)e^{in\omega_0 t} \]

We can now express our Fourier series with complex coefficients

\[ f(t) = \sum_{n=-\infty}^{\infty} C_n e^{in\omega_0 t}; C_n \in \mathbb{C} \]
Complex Fourier Coefficients

Fourier series

\[ f(t) = \sum_{n=-\infty}^{\infty} C_n e^{in\omega_0 t} \]

Fourier coefficients

\[ C_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) e^{-in\omega_0 t} dt \]

Rem (s. 39): \( C_n = |C_n| e^{i\varphi} \)  
Magnitude: \( |C_n| \)  
Phase: \( \varphi \)
Examples

Matlab demo on Fourier series

Associated to this book:
J. H. McClellan, R. W. Schafer, M. A. Yoder

Available for download here:
https://dspfirst.gatech.edu/matlab/
Fourier Transform
From Fourier Series to Transform

- \( f(t) \)
  - an aperiodic signal
  - view it as the limit of a periodic signal as \( T \to \infty \)
  - is a piecewise continuous, integrable (function space \( L^1 \)) or even a square-integrable (function space \( L^2 \)) function

\[
\int_{-\infty}^{\infty} |f(x)| \, dx < \infty \quad \int_{-\infty}^{\infty} |f(x)|^2 \, dx < \infty
\]

- For a periodic signal, the harmonic components are spaced \( \omega_0 = 2\pi/T \) apart (\( \omega_0 \) = fundamental angular frequency)

- As \( T \to \infty \), \( \omega_0 \to 0 \), and harmonic components are spaced closer and closer in frequency forming a continuous spectrum
Fourier Transform

Unitary, ordinary frequency notation

\[
\hat{f}(\xi) = \int_{-\infty}^{\infty} f(t) \cdot e^{-i2\pi \xi t} \, dt
\]

\[
f(t) = \int_{-\infty}^{\infty} \hat{f}(\xi) \cdot e^{i2\pi t \xi} \, d\xi
\]

Notes:
- \( \hat{f} \) is often replaced with \( F \)
- the direct transform is also called “analysis equation”
- the inverse transform is also called “synthesis equation”
Fourier Transform

Non-unitary, angular frequency notation

\[ F(\omega) = \int_{-\infty}^{+\infty} f(t) e^{-i\omega t} \, dt \]

\[ f(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} F(\omega) e^{i\omega t} \, d\omega \]

Notes:
• \( \omega = 2\pi \xi \rightarrow \) obtained from unitary, ordinary frequency transform with \( \xi = \omega / 2\pi \)
• \( F \) can be replaced with \( f^\wedge \)
• In electrical engineering \( i \) is substituted by \( j \) ("i" booked for current)
• Often, in order to emphasize the frequency response aspect, the imaginary aspect of the transform is emphasized: \( F(\omega), F(i\omega), \) or \( F(j\omega) \) are all equivalent notations
Common Fourier Transform

\[ f(t) = \delta(t) \]

FT (analysis equation)

IFT (synthesis equation)

\[ \hat{f}(\xi) = 1 \]
FT of a Dirac Impulse

\[ f(t) = \delta(t) \]

\[ \hat{f}(\xi) = \int_{-\infty}^{+\infty} f(t) e^{-i2\pi \xi t} \, dt = \int_{-\infty}^{+\infty} \delta(t) e^{-i2\pi \xi t} \, dt = 1 \]

\[ f(t) = \int_{-\infty}^{+\infty} \hat{f}(\xi) e^{i2\pi t \xi} \, d\xi = \int_{-\infty}^{+\infty} e^{i2\pi t \xi} \, d\xi = \delta(t) \]
Common Fourier Transform

\[ f(t) = 1 \]

\[ \hat{f}(\xi) = \delta(\xi) \]
Common Fourier Transform

\[ f(t) = \cos(2\pi at) \]

\[ \hat{f}(\xi) = \frac{\delta(\xi - a) + \delta(\xi + a)}{2} \]
Common Fourier Transform

\[ f(t) = \text{rect}(t) \]

\[ \hat{f}(\xi) = \text{sinc}(\xi) \]
Properties

Linearity
\[ h(x) = af(x) + bg(x) \Rightarrow h(\xi) = \hat{a}f(\xi) + \hat{b}g(\xi) \]

Translation
\[ h(x) = f(x - x_0) \Rightarrow h(\xi) = e^{-2\pi i x_0 \xi} \hat{f}(\xi) \]

Modulation
\[ h(x) = e^{2\pi i x_0 \xi} f(x) \Rightarrow h(\xi) = \hat{f}(\xi - \xi_0) \]

Scaling
\[ h(x) = f(ax) \Rightarrow h(\xi) = \hat{f}(\xi) \cdot \frac{1}{|a|} \]

Convolution
\[ h(x) = (f * g)(x) \Rightarrow h(\xi) = \hat{f}(\xi) \cdot \hat{g}(\xi) \]

See next lecture
Discrete Fourier Transform

\[
\hat{f}(\xi) = F(\xi) = \int_{-\infty}^{\infty} f(t) \cdot e^{-2\pi i \xi t} \, dt
\]

\[
X[k] = \sum_{n=0}^{N-1} x[n] \cdot e^{-\frac{2\pi i}{N} kn}, \quad k = 0, \cdots, N - 1
\]

- Input and output sequences are both finite (the integral becomes a finite sum)
- Both time and frequency domain are discrete (note the “[ ]”)
- Matlab, operating on a computer (i.e. a digital device) can only emulate continuity/infinity and therefore use this discrete version with an adjustable discretization level (in time, frequency, and amplitude) and finite bounds
- Also, since Matlab index start from 1, formulation is typically \( x[n+1], X[k+1] \)
Discrete Fourier Transform

\[ X[k] = \sum_{n=0}^{N-1} x[n] \cdot e^{-\frac{2\pi i}{N} kn}, \quad k = 0, \ldots, N - 1 \]

\[ x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] \cdot e^{\frac{2\pi i}{N} kn}, \quad k = 0, \ldots, N - 1 \]

- Different from the Discrete-Time Fourier Transform (DTFT), see Week 6
- Efficient implementation, for instance in Matlab, is done using a Fast Fourier Transform (FFT) algorithm (e.g., Cooley and Tukey, 1965)
- Clear correspondence with the Fourier series (see slide 41)
Fourier Tables
### Fourier Transforms Table

<table>
<thead>
<tr>
<th>Function</th>
<th>Fourier transform unitary, ordinary frequency</th>
<th>Fourier transform unitary, angular frequency</th>
<th>Fourier transform non-unitary, angular frequency</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f(x)$</td>
<td>$\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i x, \xi} dx$</td>
<td>$\hat{f}(\omega) = \int_{-\infty}^{\infty} f(x) e^{-i\omega x} dx$</td>
<td>$\hat{f}(\nu) = \int_{-\infty}^{\infty} f(x) e^{-i\nu x} dx$</td>
</tr>
<tr>
<td>$\text{rect}(ax)$</td>
<td>$\frac{1}{</td>
<td>a</td>
<td>} \cdot \text{sinc} \left( \frac{\xi}{a} \right)$</td>
</tr>
<tr>
<td>$\text{sinc}(ax)$</td>
<td>$\frac{1}{</td>
<td>a</td>
<td>} \cdot \text{rect} \left( \frac{\xi}{a} \right)$</td>
</tr>
<tr>
<td>$\text{sinc}^2(ax)$</td>
<td>$\frac{1}{</td>
<td>a</td>
<td>} \cdot \text{tri} \left( \frac{\xi}{a} \right)$</td>
</tr>
<tr>
<td>$\text{tri}(ax)$</td>
<td>$\frac{1}{</td>
<td>a</td>
<td>} \cdot \text{sinc}^2 \left( \frac{\xi}{a} \right)$</td>
</tr>
<tr>
<td>$\text{e}^{-ax} u(x)$</td>
<td>$\frac{1}{a + 2\pi i \xi}$</td>
<td>$\frac{1}{\sqrt{2\pi}(a + i\omega)}$</td>
<td>$\frac{1}{a + i\nu}$</td>
</tr>
<tr>
<td>$\text{e}^{-\alpha x^2}$</td>
<td>$\frac{\pi^{\frac{1}{2}} \cdot e^{-\frac{(x)^2}{2\alpha}}}{\sqrt{\alpha}}$</td>
<td>$\frac{1}{\sqrt{2\alpha}} \cdot e^{-\frac{\omega^2}{4\alpha}}$</td>
<td>$\frac{1}{\sqrt{\alpha}} \cdot e^{-\frac{\nu^2}{4\alpha}}$</td>
</tr>
<tr>
<td>$\text{e}^{-a</td>
<td>x</td>
<td>}$</td>
<td>$\frac{2a}{a^2 + 4\pi^2 \xi^2}$</td>
</tr>
</tbody>
</table>

Note $\nu = \omega$ on s. 46

Simple example s. 58
# Fourier Properties Table

<table>
<thead>
<tr>
<th>Function</th>
<th>Fourier transform unitary, ordinary frequency</th>
<th>Fourier transform unitary, angular frequency</th>
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</tr>
</thead>
<tbody>
<tr>
<td>$f(x)$</td>
<td>$\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i x \xi} dx$</td>
<td>$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\omega x} dx$</td>
<td>$\hat{f}(\nu) = \int_{-\infty}^{\infty} f(x) e^{-i\nu x} dx$</td>
</tr>
<tr>
<td>$a \cdot f(x) + b \cdot g(x)$</td>
<td>$a \cdot \hat{f}(\xi) + b \cdot \hat{g}(\xi)$</td>
<td>$a \cdot \hat{f}(\omega) + b \cdot \hat{g}(\omega)$</td>
<td>$a \cdot \hat{f}(\nu) + b \cdot \hat{g}(\nu)$</td>
</tr>
<tr>
<td>$f(x - a)$</td>
<td>$e^{-2\pi i a \xi} \hat{f}(\xi)$</td>
<td>$e^{-i\omega a} \hat{f}(\omega)$</td>
<td>$e^{-i\nu a} \hat{f}(\nu)$</td>
</tr>
<tr>
<td>$e^{2\pi i a x} f(x)$</td>
<td>$\hat{f}(\xi - a)$</td>
<td>$\hat{f}(\omega - 2\pi a)$</td>
<td>$\hat{f}(\nu - 2\pi a)$</td>
</tr>
<tr>
<td>$f(\alpha x)$</td>
<td>$\frac{1}{</td>
<td>\alpha</td>
<td>} \hat{f} \left( \frac{\xi}{\alpha} \right)$</td>
</tr>
<tr>
<td>$f(x)$</td>
<td>$f(-\xi)$</td>
<td>$f(-\omega)$</td>
<td>$2\pi f(-\nu)$</td>
</tr>
<tr>
<td>$\frac{d^n f(x)}{dx^n}$</td>
<td>$(2\pi i \xi)^n \hat{f}(\xi)$</td>
<td>$(i\omega)^n \hat{f}(\omega)$</td>
<td>$(i\nu)^n \hat{f}(\nu)$</td>
</tr>
<tr>
<td>$x^n f(x)$</td>
<td>$\left( \frac{i}{2\pi} \right)^n \frac{d^n \hat{f}(\xi)}{d\xi^n}$</td>
<td>$\frac{i^n}{\omega^n} \frac{d^n \hat{f}(\omega)}{d\omega^n}$</td>
<td>$\frac{i^n}{\nu^n} \frac{d^n \hat{f}(\nu)}{d\nu^n}$</td>
</tr>
<tr>
<td>$(f * g)(x)$</td>
<td>$\hat{f}(\xi) \hat{g}(\xi)$</td>
<td>$\sqrt{2\pi} \hat{f}(\omega) \hat{g}(\omega)$</td>
<td>$\hat{f}(\nu) \hat{g}(\nu)$</td>
</tr>
<tr>
<td>$f(x)g(x)$</td>
<td>$(\hat{f} \ast \hat{g})(\xi)$</td>
<td>$\left( \frac{\hat{f} \ast \hat{g}}{\sqrt{2\pi}} \right)(\omega)$</td>
<td>$\frac{1}{2\pi} (\hat{f} \ast \hat{g})(\nu)$</td>
</tr>
</tbody>
</table>

Combined example s. 59
Simple Example

\[ f(t) = \text{trian}(t) \]

\[ \hat{f}(\xi) = \text{sinc}^2(\xi) \]
Combined Example

\[ f(t) = \cos(2\pi(3t))e^{-\pi t^2} \]

The Fourier transform of \( f(t) \)
Conclusion
Take Home Messages

• A signal can be a varying quantity in time and/or space
• Signal classification: continuous vs. discrete in time, space and amplitude
• Fourier decomposition: every periodic signal can be decomposed into a sum of sines and cosines
• Fourier coefficients are the weights in this sum
• Fourier Transform (FT) for aperiodic signals
• Implementation on computers of Fourier transform is discretized and bound: Discrete Fourier Transform (DFT)
• The Fast FT (FFT) is a fast algorithm for solving the DFT
• Properties of FT and tables help to solve analytically analysis and synthesis equations
Additional Literature – Week 4

Related course material
- Information, Calcul, Communication (ICC)
- Analysis IV
- Quantitative Methods II

Books
- J. H. McClellan, R. W. Schafer, M. A. Yoder