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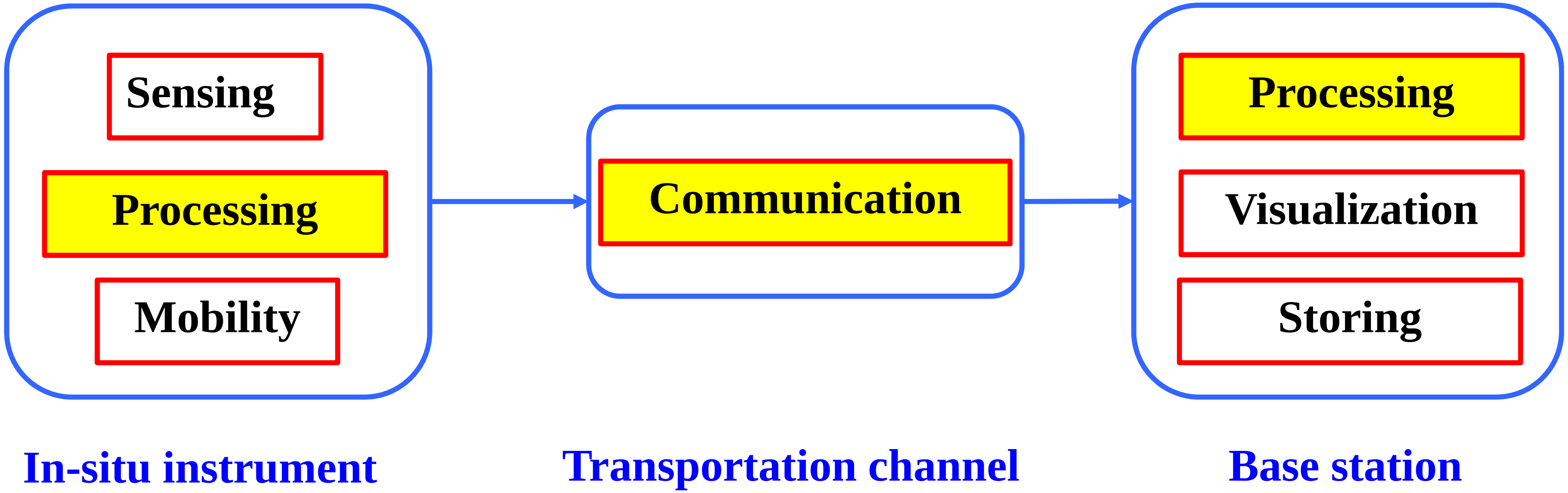
FÉDÉRALE DE LAUSANNE

Signals, Instruments, and Systems – W5

Introduction to Signal Processing – Convolution, Sampling, Reconstruction

Motivation from Week 1 Lecture



Highlighted blocks are those mainly leveraging the content of this lecture.

Convolution

Convolution

$$(f * g)(t) = \int_{-\infty}^{\infty} f(\tau) \cdot g(t-\tau) d\tau$$

- For each value of *t*:
  1. Flip (reflect) ***g***
  2. Shift ***g*** by ***t***
  3. Multiply ***f*** and ***g***
  4. Integrate over ***τ***

1) $g(\tau) \rightarrow g(-\tau)$

2) $g(-\tau) \rightarrow g(t-\tau)$

3) $f(\tau) \cdot g(t-\tau)$

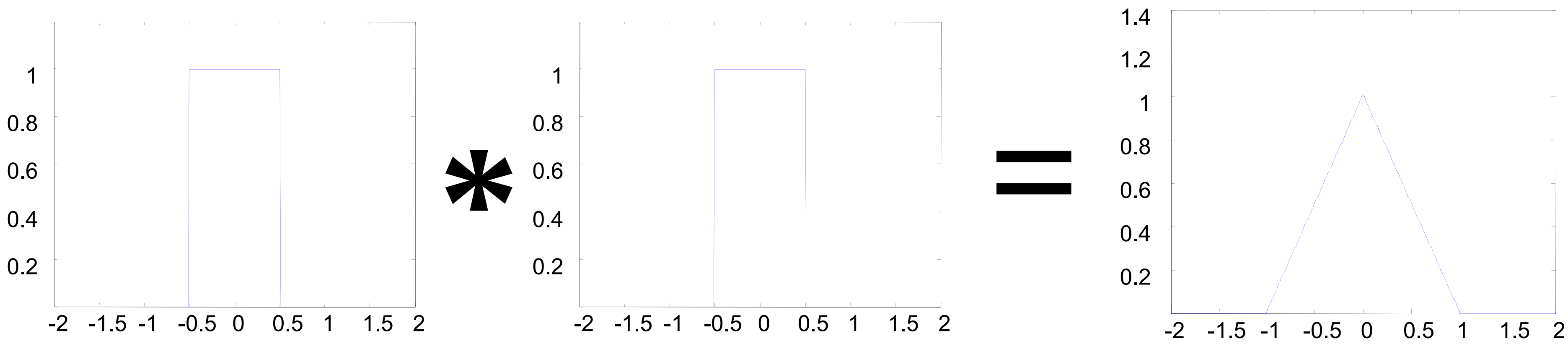
4) $\int_{-\infty}^{\infty} f(\tau) \cdot g(t-\tau) d\tau$

- Note that the result does **not** depend on τ!

[Matlab demo "cconvdemo"]

http://users.ece.gatech.edu/mcclella/matlabGUIs/

Examples



1
0.8
0.6
0.4
0.2
-2 -1.5 -1 -0.5 0 0.5 1 1.5 2
*
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-2 -1.5 -1 -0.5 0 0.5 1 1.5 2
=
1.4
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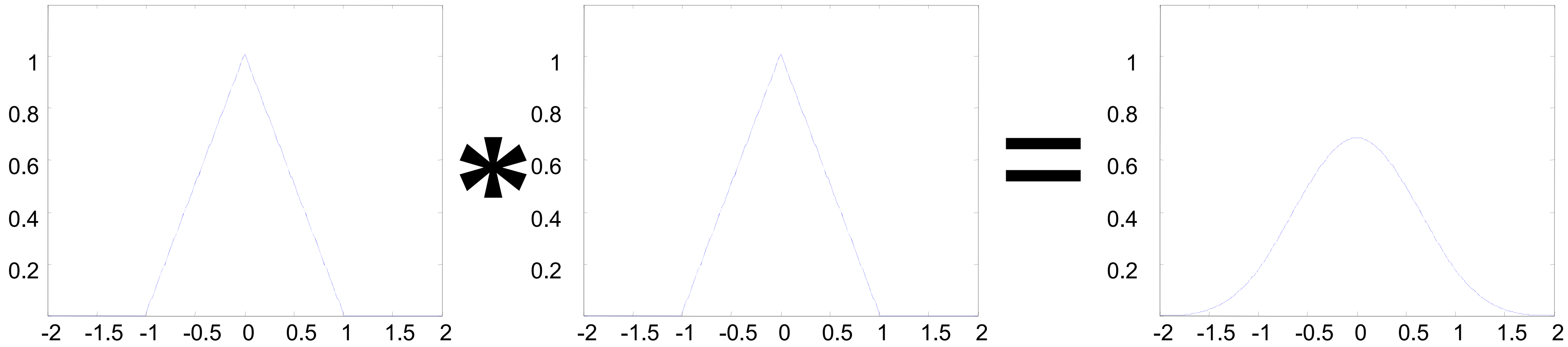
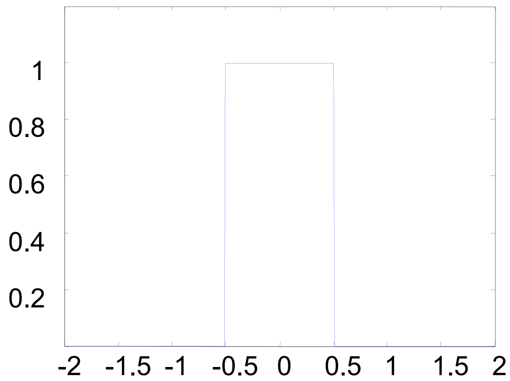
-2 -1.5 -1 -0.5 0 0.5 1 1.5 2

Examples



Examples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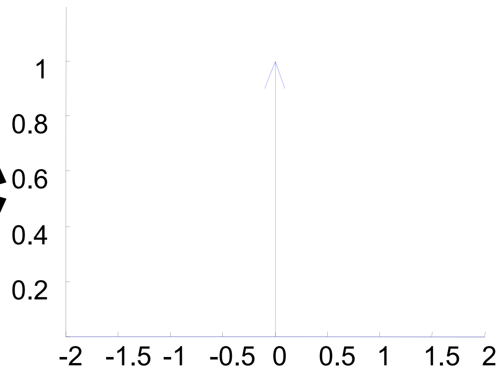
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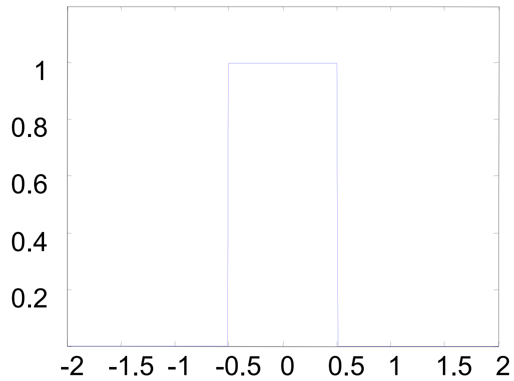
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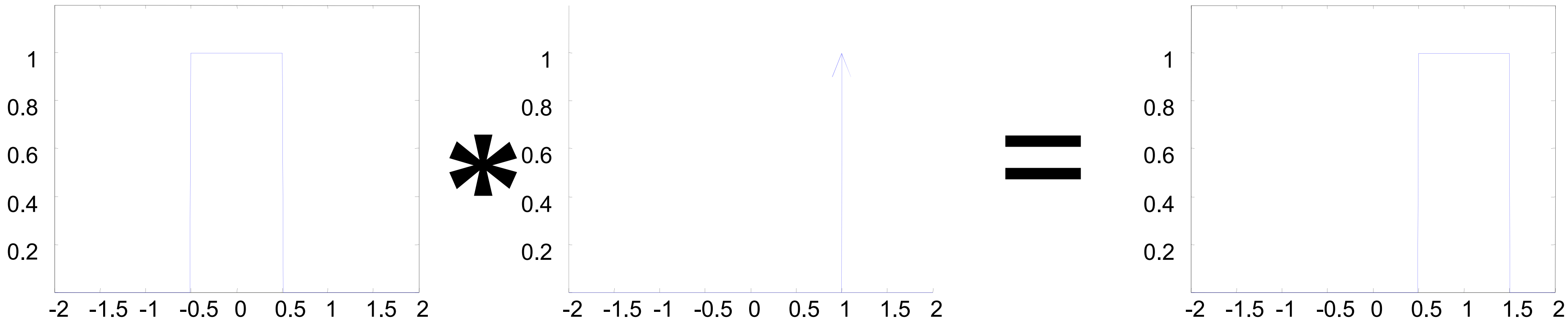
Examples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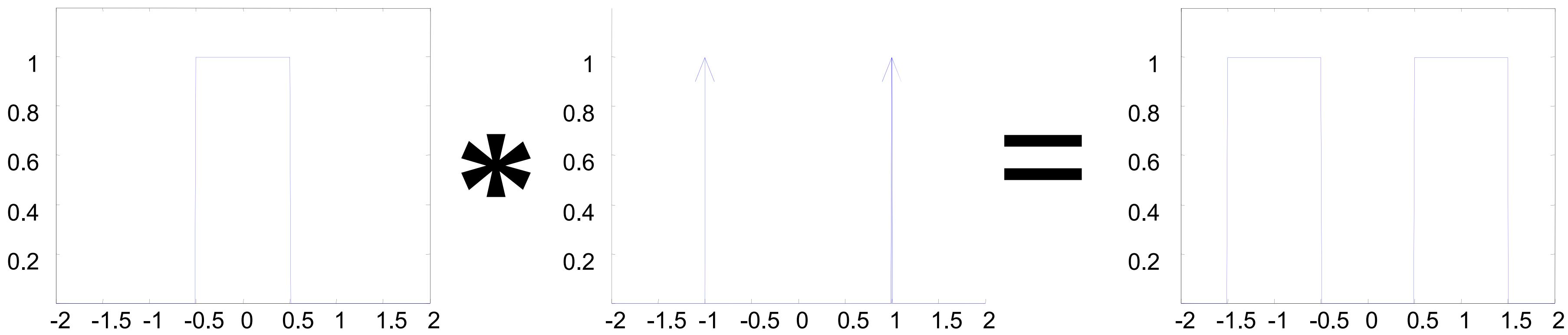
Examples



Convolution in Time and Frequency Domains

Time domain | **Frequency domain**

$$h(t) = (f * g)(t) \Leftrightarrow \hat{h}(\xi) = \hat{f}(\xi) \cdot \hat{g}(\xi)$$

$$h(t) = f(t) \cdot g(t) \Leftrightarrow \hat{h}(\xi) = (\hat{f} * \hat{g})(\xi)$$

Unitary, ordinary frequency

$$h(t) = (f * g)(t) \Leftrightarrow H(\omega) = F(\omega) \cdot G(\omega)$$

$$h(t) = f(t) \cdot g(t) \Leftrightarrow H(\omega) = \frac{1}{2\pi}(F * G)(\omega)$$

Non-unitary, angular frequency

Convolution Properties

Commutativity

$$f * g = g * f$$

Associativity

$$f * (g * h) = (f * g) * h$$

Distributivity

$$f * (g + h) = (f * g) + (f * h)$$

Associativity with scalar multiplication

$$a(f * g) = (af) * g = f * (ag)$$

Discrete Convolution

$$(f * g)(t) = \int_{-\infty}^{\infty} f(\tau) \cdot g(t-\tau) d\tau$$

$$\downarrow$$

$$(f * g)[n] = \sum_{m=-\infty}^{\infty} f[m] \cdot g[n-m]$$

Notes:

- Similar to the continuous version
- The integral becomes an infinite sum (note: in theory no bounds on the sum by definition as in the DFT)
- In practice, a computer (i.e. a digital device), running Matlab for instance, can only emulate continuity and therefore discrete time and quantized amplitude as well as finite bounds of the convolution window are used

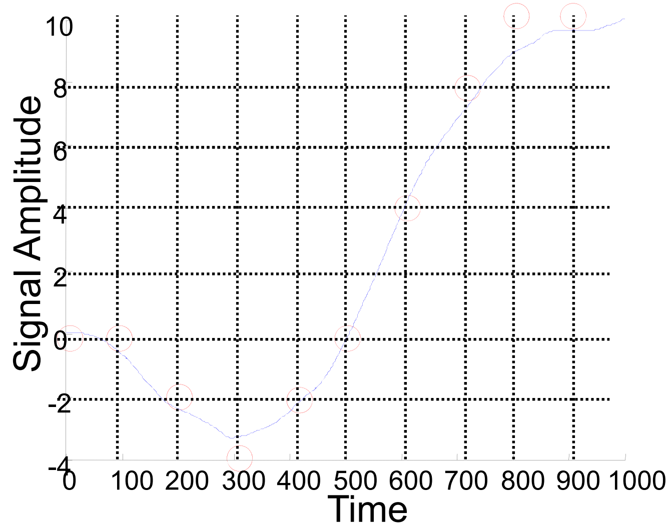
[Matlab demo “dconvdemo”]

http://users.ece.gatech.edu/mcclella/matlabGUIs/

Sampling

Analog-Digital Converter (ADC)

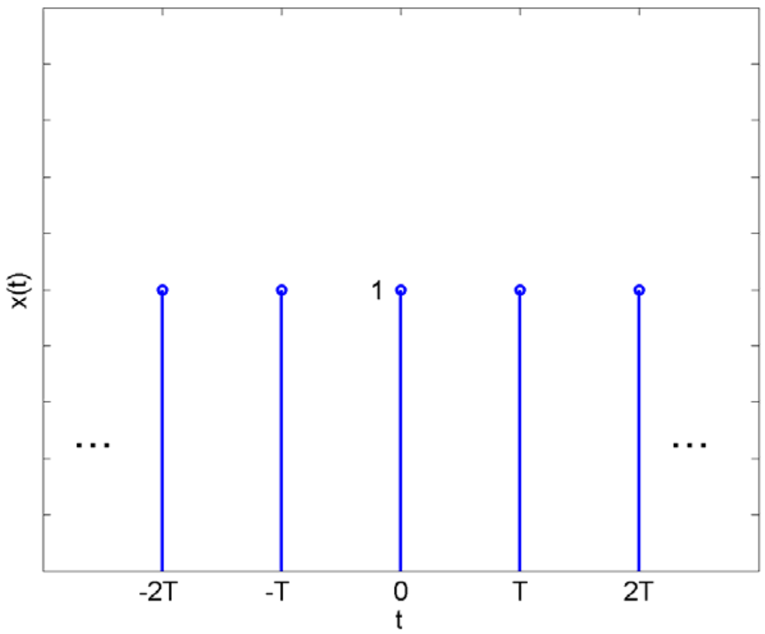
- Transforms continuous analog signal into series of values
- Two key elements
  - **Sampling** (in time)
  - **Quantization** (of values)



y[n]=0 0 -2 -4 -2 0 4 8 10 10

Periodic Train of Dirac Impulses

$$x(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$$



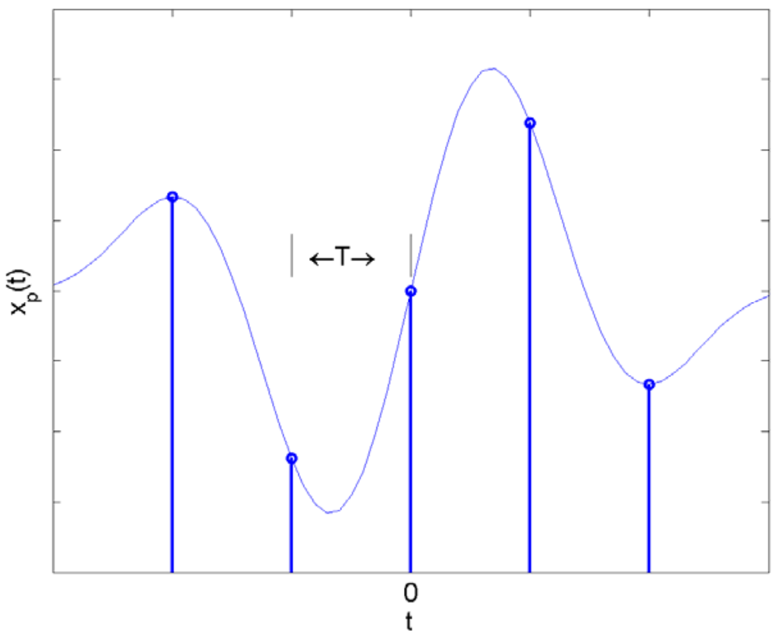
Sampling in Time Domain

$$x(t)$$

$$p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

Periodic train of Dirac pulses

$$x_p(t) = x(t)p(t) = \sum_{n=-\infty}^{\infty} x(t)\delta(t - nT) = \sum_{n=-\infty}^{\infty} x(nT)\delta(t - nT)$$



T = sampling period
f_s = 1/T = sampling frequency

Sampling in the Frequency Domain

- It would be nice to understand what does it mean sampling in the frequency domain so that we can leverage this representation for further reasoning (e.g., filter design)
- Multiplication in time domain means convolution in frequency domain but …
- how does look like a periodic train of Dirac impulses in the frequency domain?

From W4 (s.41): Fourier Series with Complex Fourier Coefficients

Fourier series $$f(t) = \sum_{n=-\infty}^{\infty} C_n e^{in\omega_0 t}$$

Fourier coefficients $$C_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) e^{-in\omega_0 t} dt$$

Rem: $C_n = |C_n| e^{i\varphi}$ Magnitude: $|C_n|$ Phase: φ

From W4 (s.46): Fourier Transform

Non-unitary, angular frequency notation

$$F(\omega) = \int_{-\infty}^{+\infty} f(t) e^{-i\omega t} dt$$

Fourier Transform

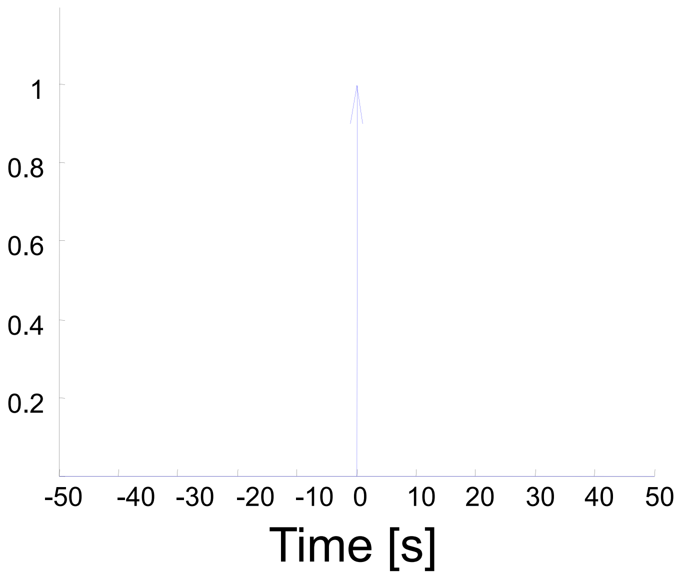
$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} F(\omega) e^{i\omega t} d\omega$$

Inverse Fourier Transform

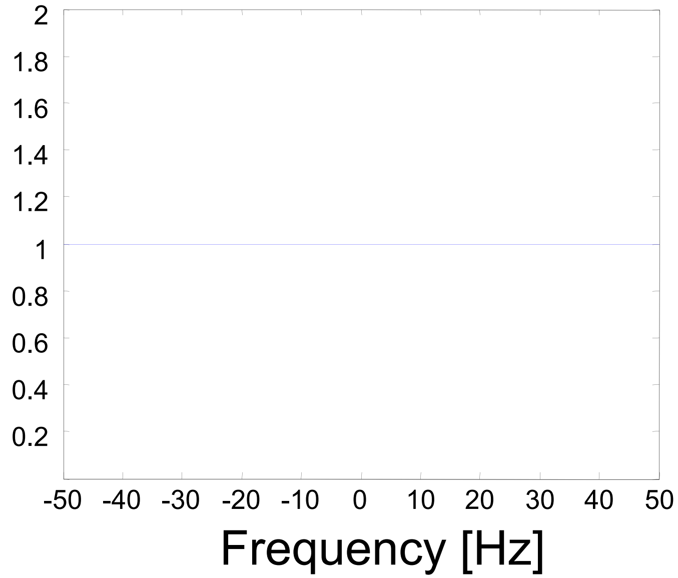
Notes:

- $\omega = 2\pi\xi \rightarrow$ obtained from unitary, ordinary frequency transform with $\xi = \omega/2\pi$
- F can be replaced with f^
- In electrical engineering *i* is substituted by *j* ("i" booked for current)
- Often, in order to emphasize the frequency response aspect, the imaginary aspect of the transform is emphasized: F(ω), F(iω), or F(jω) are all equivalent notations

From W4 (s. 47): FT of Dirac Impulse



$$f(t) = \delta(t)$$



$$\hat{f}(\xi) = 1$$

FT for Periodic Signals

- Although it generalizes to aperiodic signals, the FT can be also applied to periodic signals
- We can derive the FT of a periodic signal from its Fourier series (which have been developed for periodic signals, see W4)
- The FT of a periodic Dirac train of impulses consists of a periodic train of impulses in the frequency domain as well, with the area of impulses proportional to the Fourier series coefficients

From [Oppenheim et al., 1997]

FT for Periodic Signals

(assume: period T, $\omega_0 = 2\pi/T$)

$$X(\omega) = 2\pi\delta(\omega - \omega_0)$$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} 2\pi\delta(\omega - \omega_0)\, e^{i\omega t} d\omega$$

W4, s. 46, IFT

$$x(t) = e^{i\omega_0 t}$$

W4, s. 52, modulation property

Linear combination of impulses equally spaced in frequency:

$$X(\omega) = \sum_{n=-\infty}^{+\infty} 2\pi C_n \delta(\omega - n\omega_0)$$

$$x(t) = \sum_{n=-\infty}^{+\infty} C_n e^{in\omega_0 t}$$

Compare with W4, s. 41, Fourier series

FT for Periodic Impulse Train

(assume: period T, $\omega_0 = 2\pi/T$)

Now consider the periodic impulse train of before:

$$x(t) = \sum_{n=-\infty}^{+\infty} \delta(t - nT)$$

Calculate the coefficient of its Fourier series:

$$C_n = \frac{1}{T}\int_{-T/2}^{+T/2} \delta(t) e^{-in\omega_0 t} dt = \frac{1}{T}$$

Compare with
W4, s. 41, Fourier series
T= 2π for sin/cos functions

This means that each Fourier coefficient of the periodic impulse train has the same value; insert C_n in previous expression (s. 24):

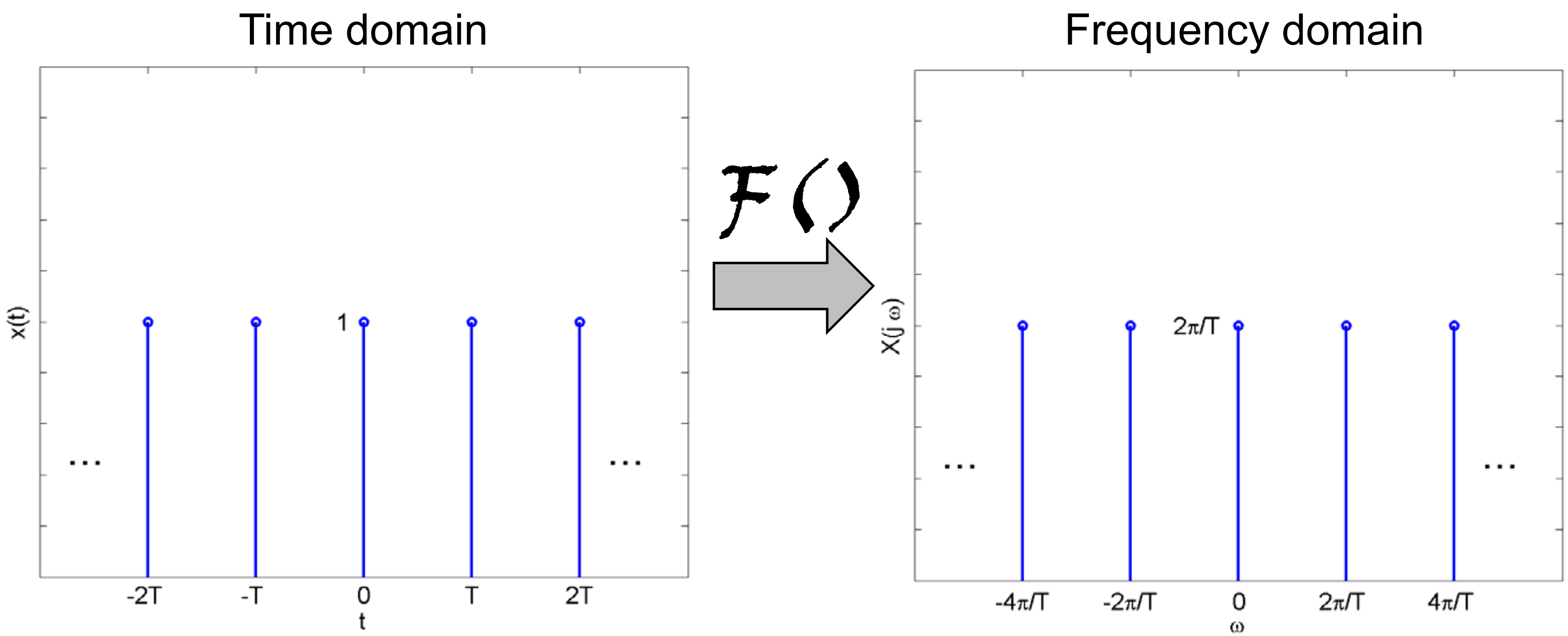
$$X(\omega) = \sum_{n=-\infty}^{+\infty} 2\pi C_n \delta(\omega - n\omega_0)$$

$$X(\omega) = \sum_{n=-\infty}^{+\infty} \frac{2\pi}{T} \delta(\omega - n\omega_0) = \frac{2\pi}{T} \sum_{n=-\infty}^{+\infty} \delta(\omega - n\omega_0)$$

Train of Dirac Impulses

$$x(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

$$X(\omega) = \sum_{k=-\infty}^{\infty} \frac{2\pi}{T} \delta\left(\omega - \frac{k2\pi}{T}\right)$$



Sampling in Frequency Domain

Time domain $x_p(t) = x(t)p(t)$

Frequency domain $X_p(\omega) = \frac{1}{2\pi} X(\omega) * P(\omega)$

$$P(\omega) = \frac{2\pi}{T} \sum_{k=-\infty}^{\infty} \delta(\omega - k\omega_S)$$

$\omega_s = \frac{2\pi}{T}$ Sampling angular frequency

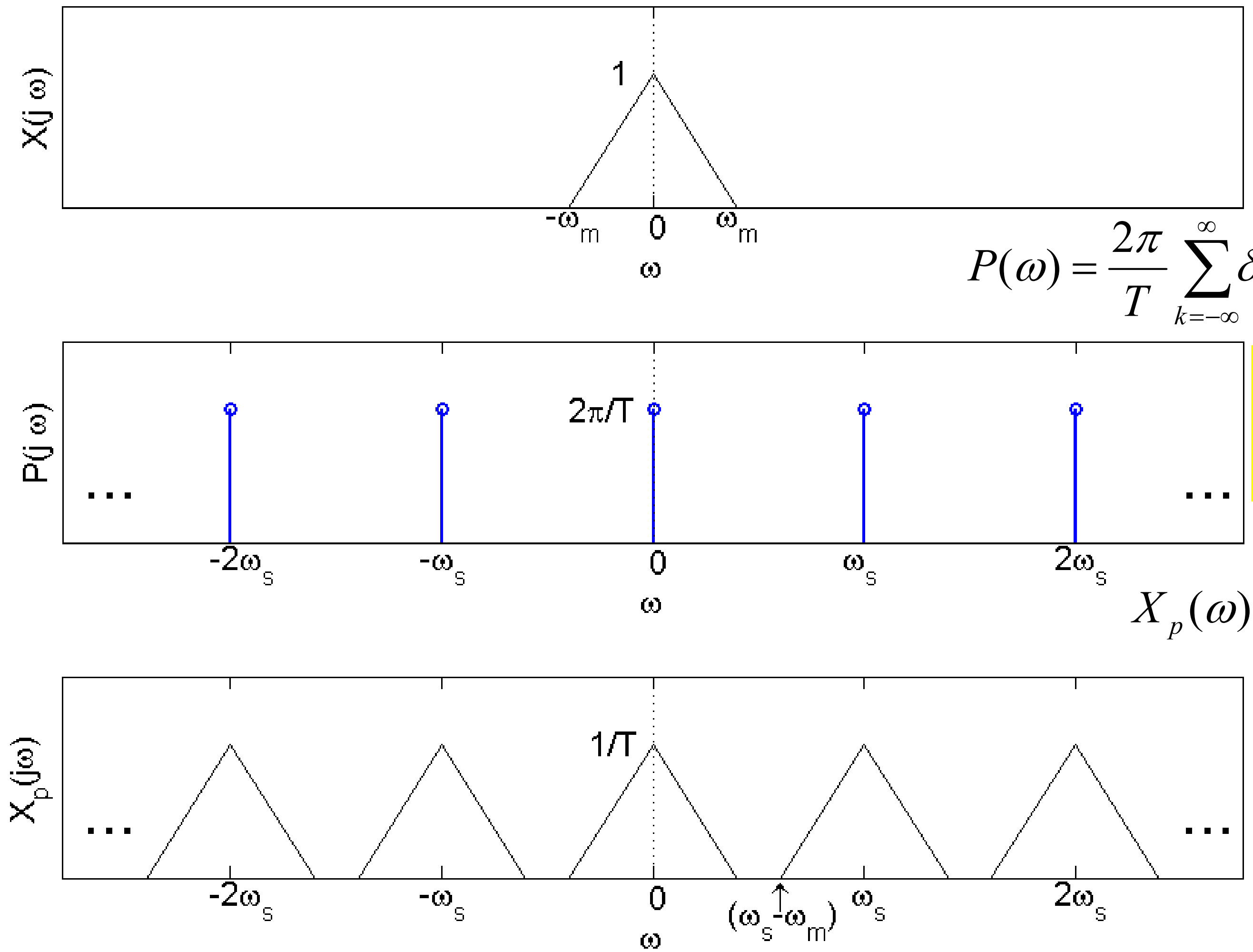
$$X_p(\omega) = \frac{1}{T} \sum_{k=-\infty}^{\infty} X(\omega) * \delta(\omega - k\omega_S)$$

$$= \frac{1}{T} \sum_{k=-\infty}^{\infty} X(\omega - k\omega_S)$$

Note: see also ss. 8-10 as examples for this operation

Sampling a Band-Limited Signal

X(ω) → spectrum of signal x(t) with highest frequency < ω_m



$$P(\omega) = \frac{2\pi}{T} \sum_{k=-\infty}^{\infty} \delta(\omega - k\omega_S)$$

Angular sampling frequency

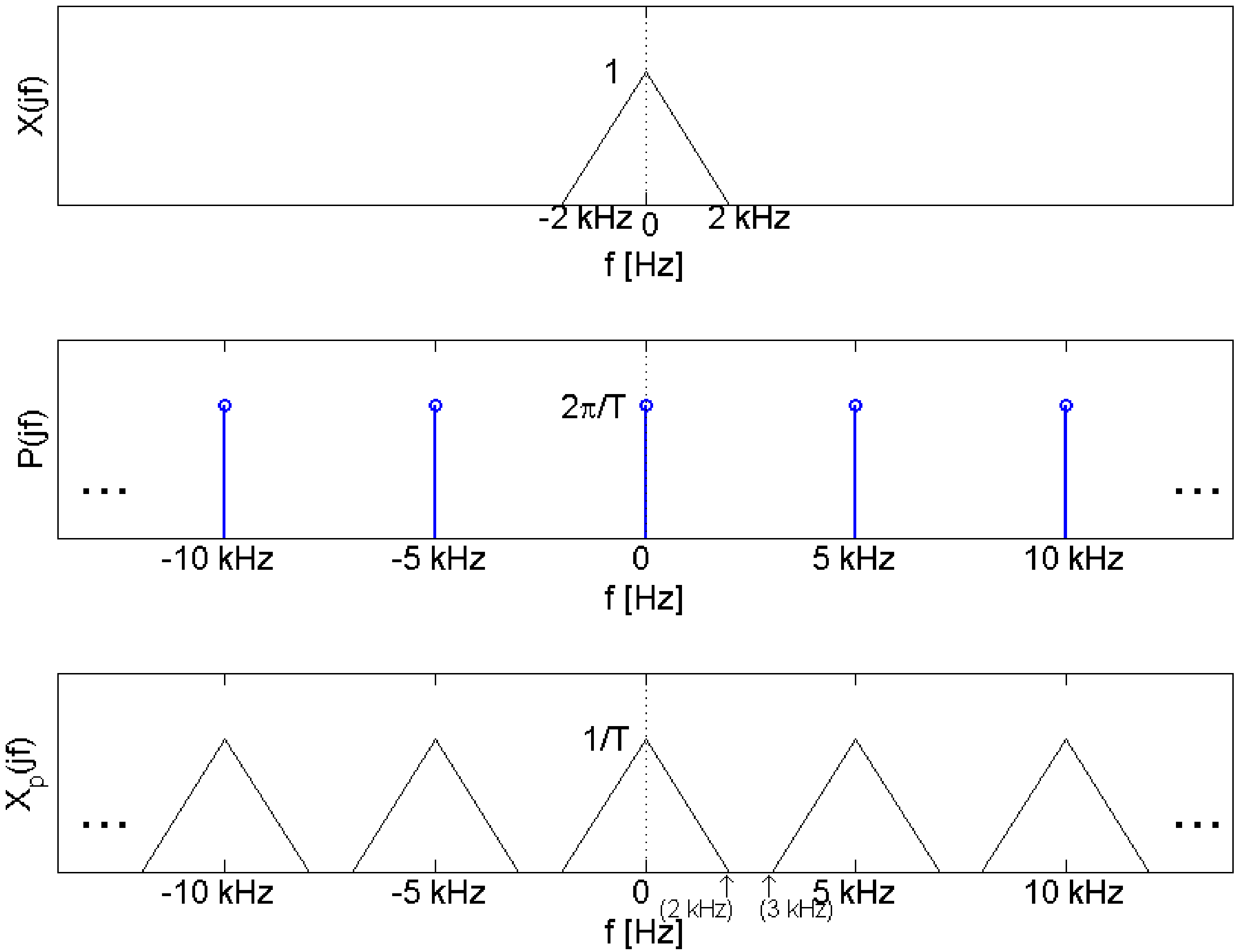
$\omega_s > 2\,\omega_m$

$$X_p(\omega) = \frac{1}{2\pi} X(\omega) * P(\omega)$$

$$= \frac{1}{T} \sum_{k=-\infty}^{\infty} X(\omega - k\omega_S)$$

Sampling a Band-Limited Signal

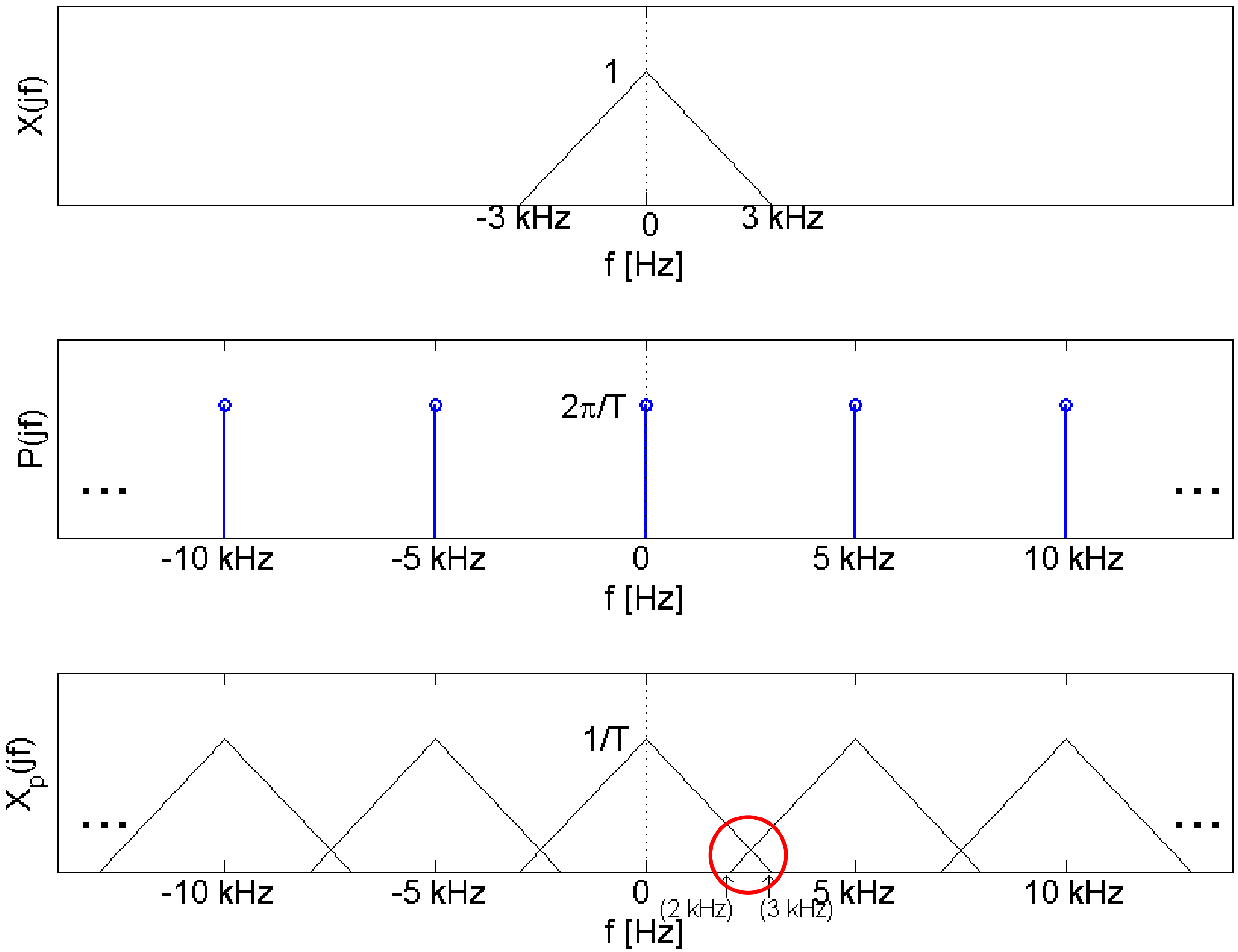
X(ξ) → spectrum of signal x(t)
with highest frequency < 2 kHz



Sampling frequency:
5 kHz > 2 * 2 kHz

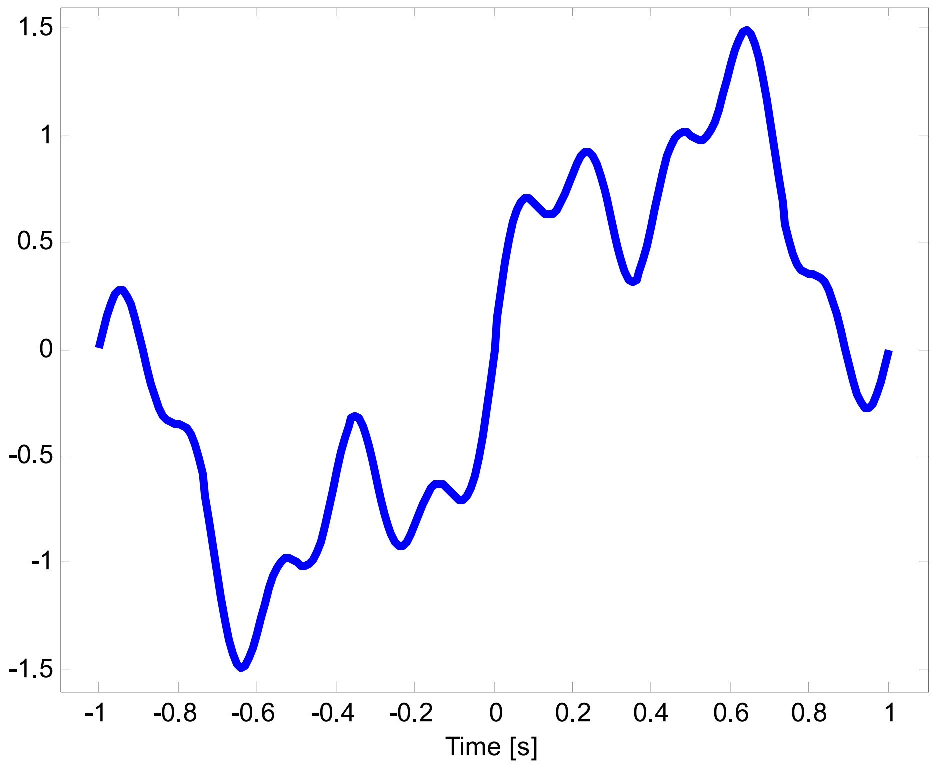
Sampling a Band-Limited Signal

X(ξ) → spectrum of signal x(t)
with highest frequency < 3 kHz



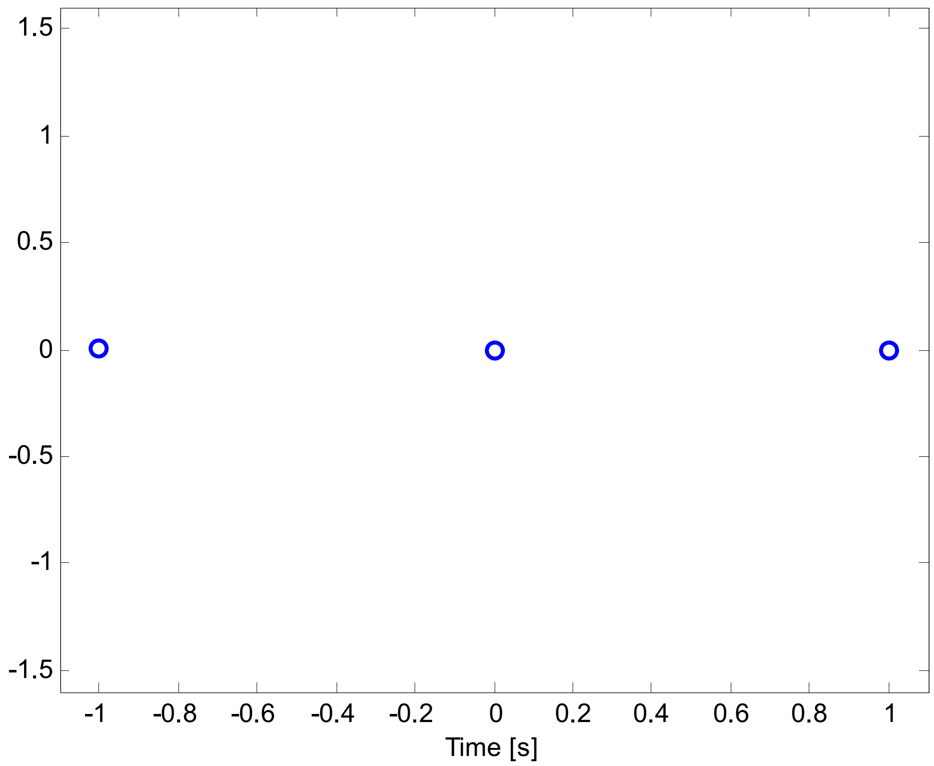
Sampling frequency:
5 kHz < 2 * 3 kHz

Original Signal



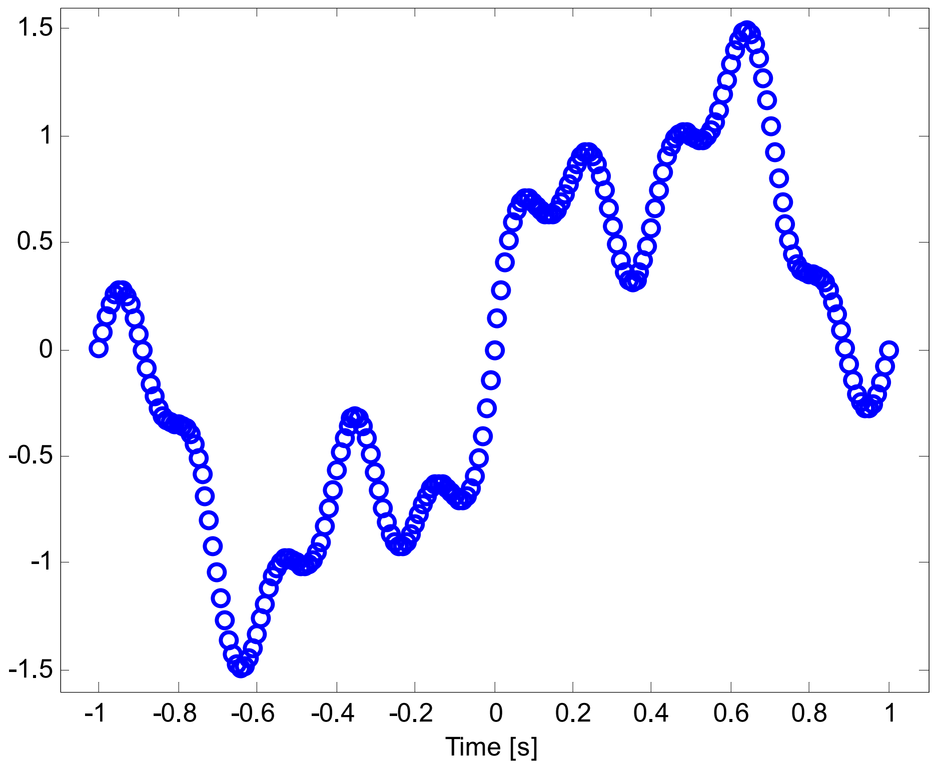
$$f(t) = \sin(2\pi t) + 0.4\sin(2\pi \cdot 2t) + 0.2\sin(2\pi \cdot 5t)$$

Too Few Samples (1Hz)



→ Data is lost

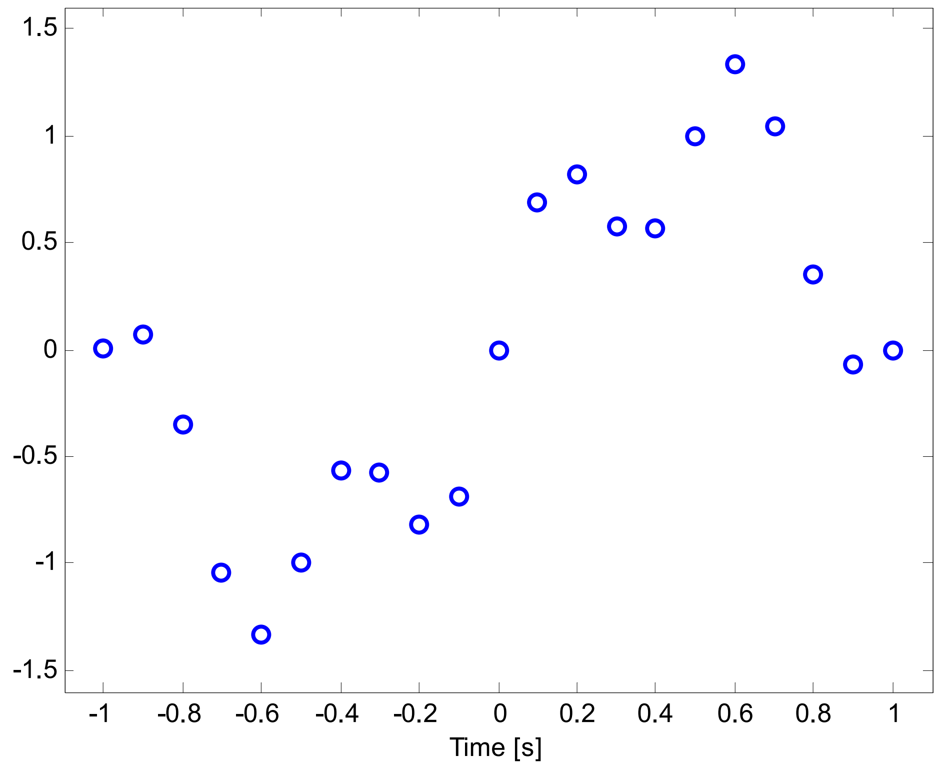
Too Many Samples (100 Hz)



→Redundant data

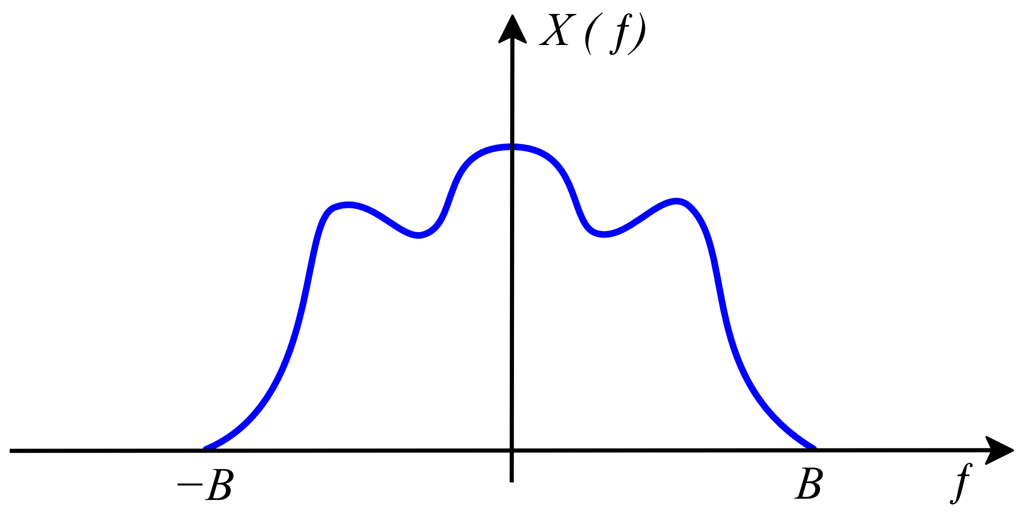
→Increase of data size

Minimal Possible Sampling (> 10 Hz)



Nyquist–Shannon Theorem

- If a function $x(t)$ contains no frequencies higher than B Hz, it is completely determined by giving its coordinates at a series of points spaced $1/(2B)$ seconds apart.
- Sampling frequency must be at least two times greater than the maximal signal frequency



Sampling in Practice

- Sampling frequency two times greater than maximal frequency is the limit
- Example: audio CD, sampling at 44.1 kHz since maximal hearable frequency: 20 kHz
- If possible, try to use a sampling frequency 10 times greater than the maximal frequency (help all sorts of filtering and reconstruction processes)

Signal Reconstruction

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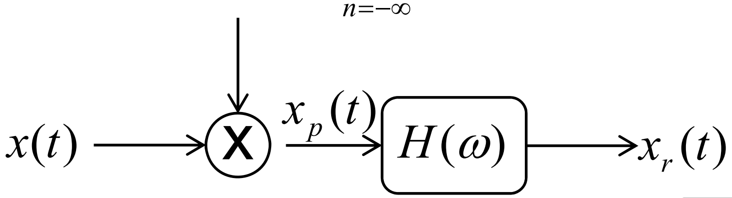
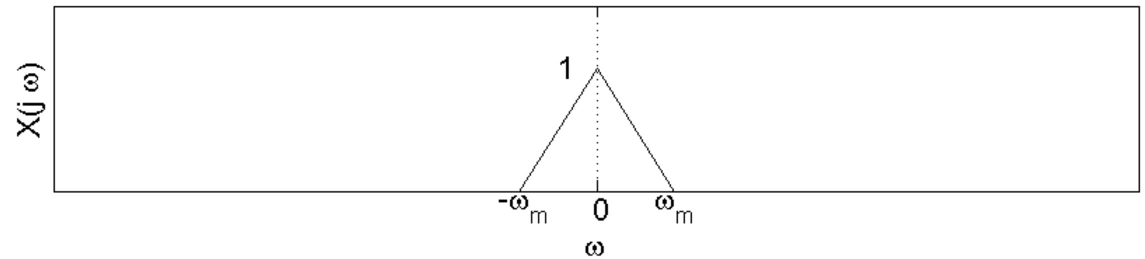
$$p(t) = \sum_{n=-\infty}^{+\infty} \delta(t - nT)$$



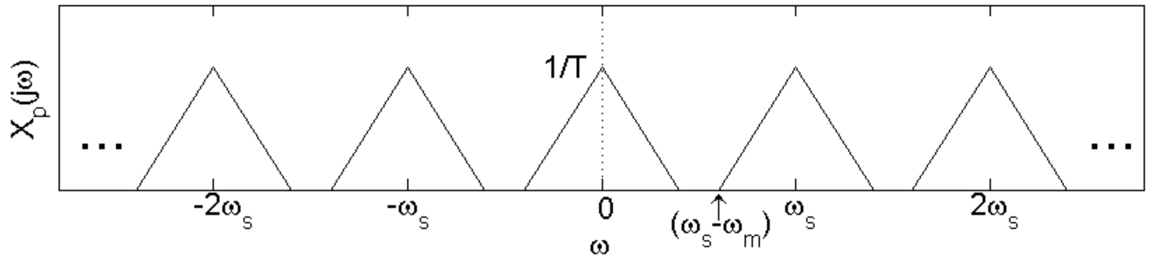
If there is no overlap between the shifted spectra the signal $x_r(t)$ can be perfectly reconstructed from x(t)

spectrum of original signal



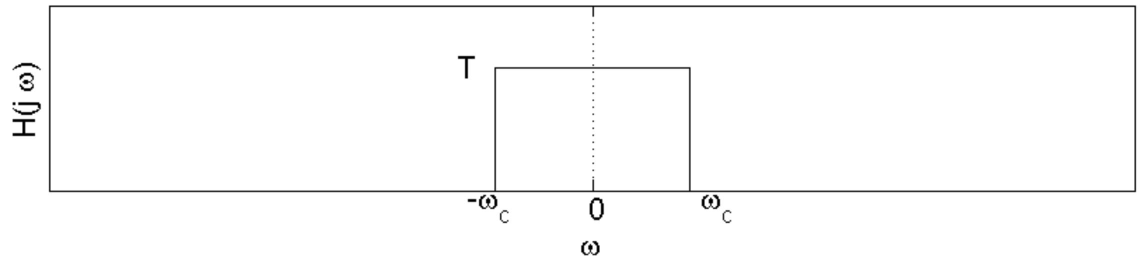
spectrum of sampled signal

sampling angular frequency
$\omega_s > 2\,\omega_m$



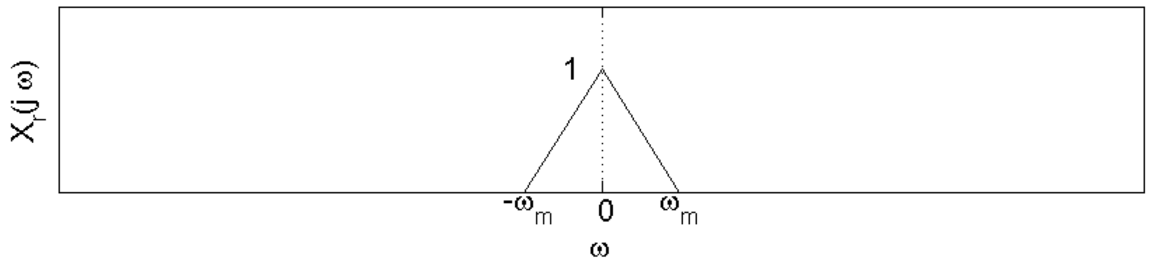
filtering

filter cut-off angular frequency
$\omega_m < \omega_c < (\omega_s - \omega_m)$

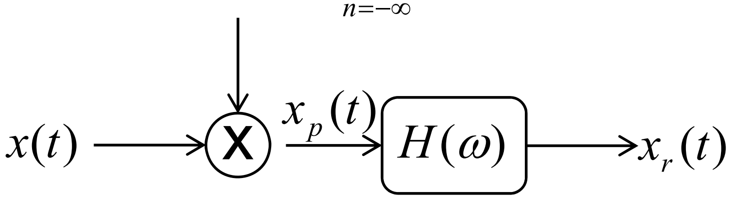


$$X_r(\omega) = X_p(\omega) H(\omega)$$

spectrum of reconstructed signal

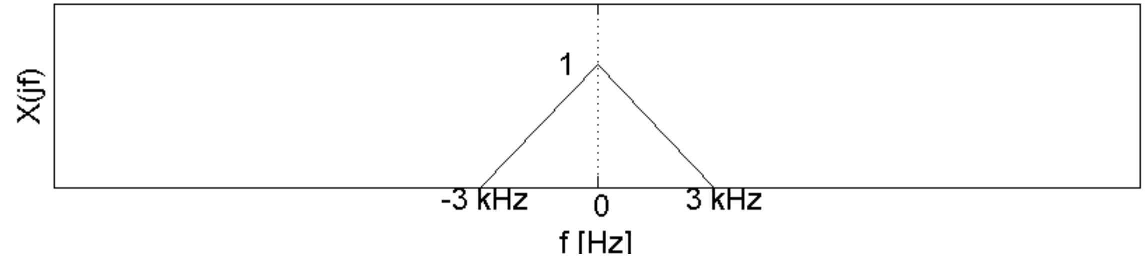


$$p(t) = \sum_{n=-\infty}^{+\infty} \delta(t - nT)$$



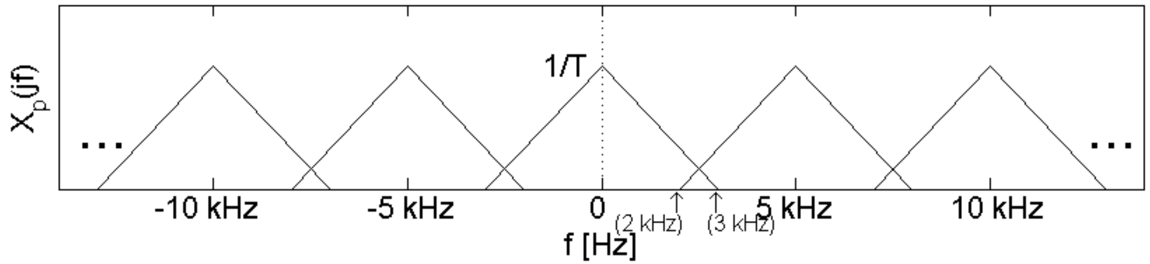
If there is overlap between the shifted spectra the signal $x_r(t)$ cannot be perfectly reconstructed from x(t)

spectrum of original signal



spectrum of sampled signal

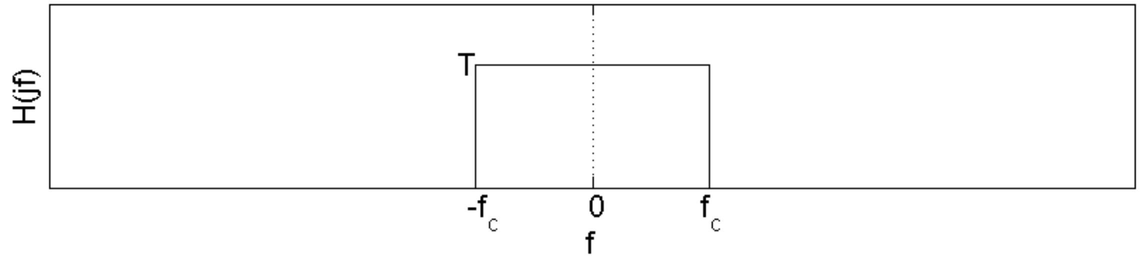
sampling frequency
$f_s < 2\, f_m$



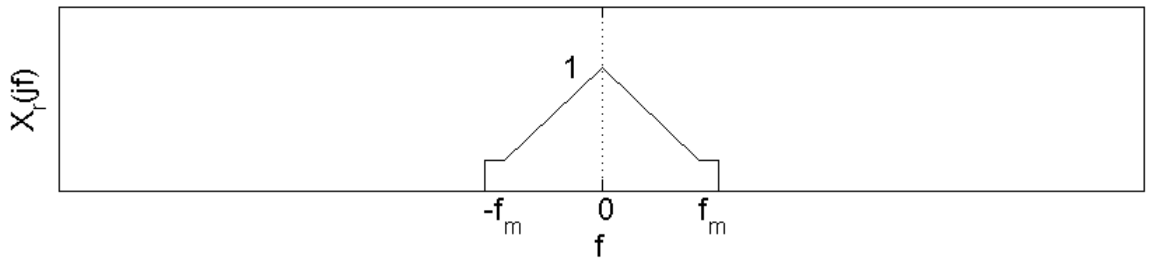
filtering

filter cut-off frequency f_c
$f_m < f_c < (f_s - f_m)$

$$X_r(\omega) = X_p(\omega) H(\omega)$$

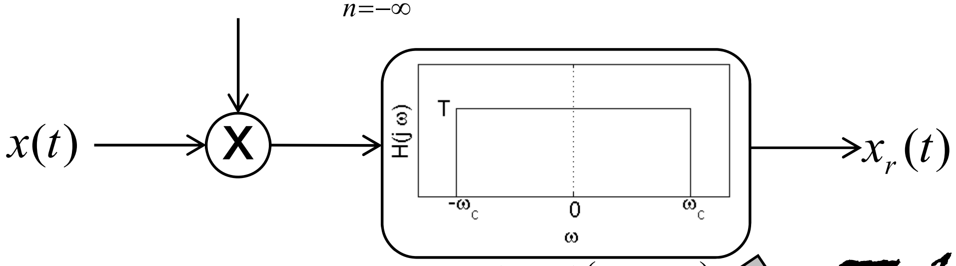


spectrum of reconstructed signal



Time Domain Interpretation of Signal Reconstruction

$$p(t) = \sum_{n=-\infty}^{+\infty} \delta(t - nT)$$



$\omega_m < \omega_c < (\omega_s - \omega_m)$

$\mathcal{F}^{-1}()$

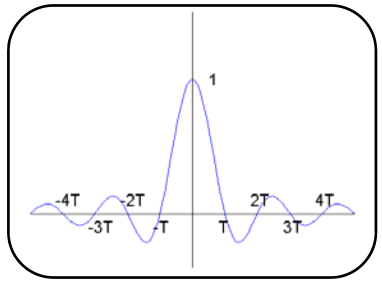
$$x_r(t) = x_p(t) * h(t)$$

$$= \left(\sum_{n=-\infty}^{\infty} x(nT)\delta(t - nT) \right) * h(t)$$

$$= \sum_{n=-\infty}^{\infty} x(nT) h(t - nT)$$

$$= \sum_{n=-\infty}^{\infty} x(nT) \frac{T \sin(\omega_c (t - nT))}{\pi (t - nT)}$$

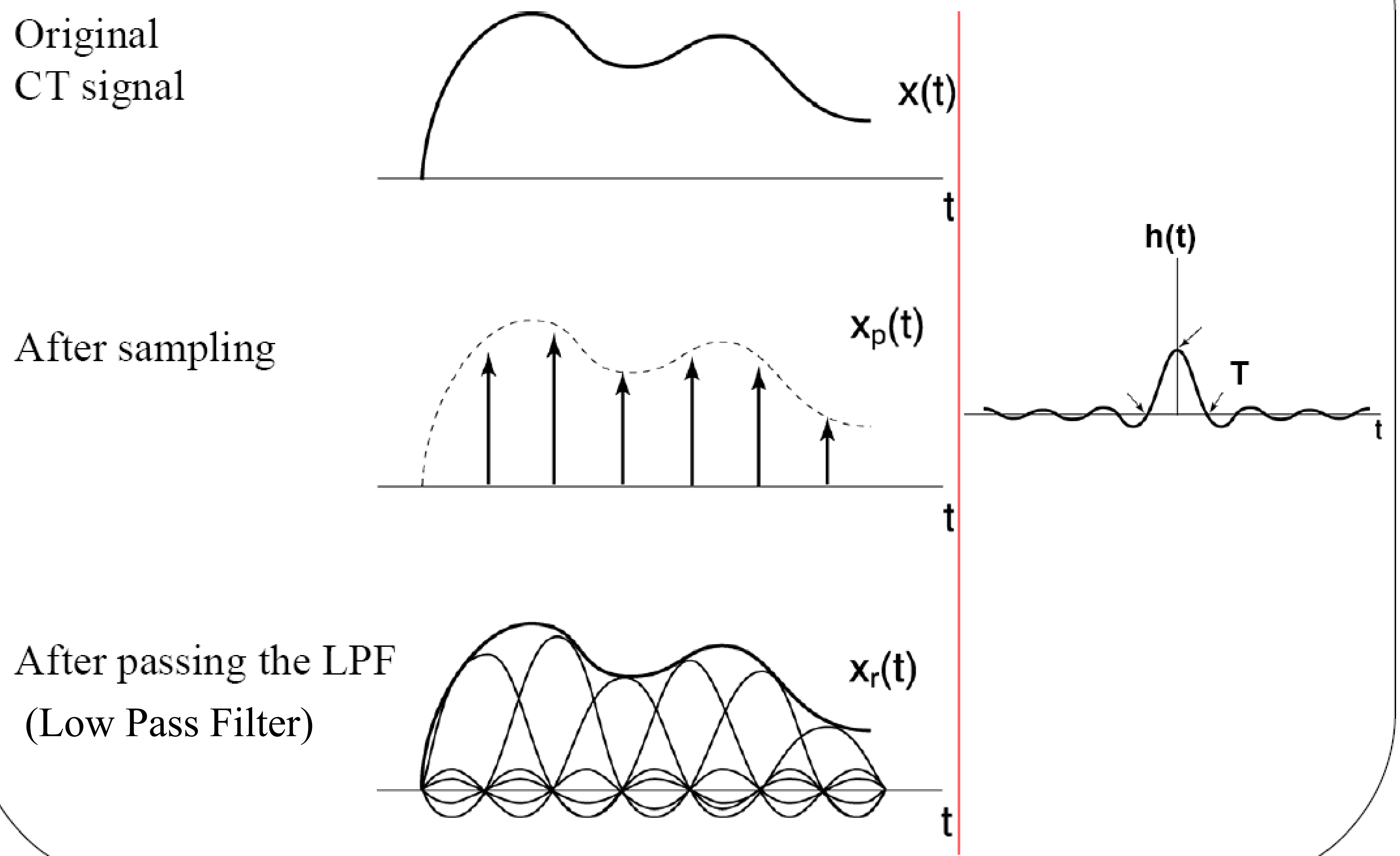
with $h(t) = \dfrac{T \sin(\omega_c t)}{\pi t}$



The lowpass filter interpolates the samples assuming x(t) contains no energy at frequencies > ω_c

(ω_c =cutoff angular frequency)

Graphic Illustration of Time-Domain Interpolation



From Prof. A. S. Willsky, Signals and Systems course

Signal Reconstruction in Practice

1. **Whittaker-Shannon interpolation** (band-limited interpolation):

$$x_r(t) = \sum_{n=-\infty}^{\infty} x(nT) \cdot \mathrm{sinc}\left(\frac{t-nT}{T}\right)$$

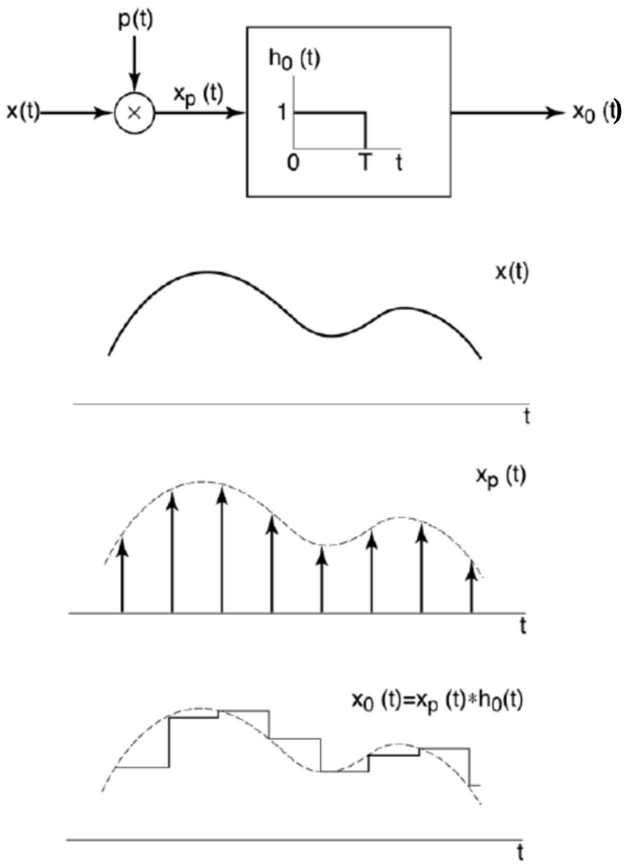
$$x_r(t) = \left(\sum_{n=-\infty}^{\infty} x(nT) \cdot \delta(t-nT)\right) * \mathrm{sinc}\left(\frac{t}{T}\right)$$ (Alternative equivalent formulation)

- Signal has to be band limited
  (i.e. Fourier transform for frequencies greater than B equal 0)
- The sampling rate must exceed twice the bandwidth, 2B

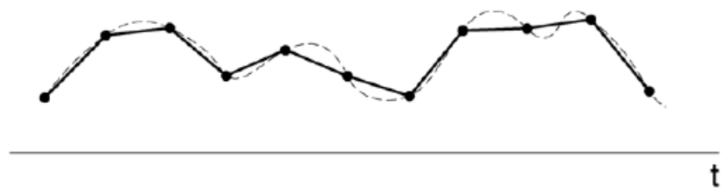
$$f_s > 2B \qquad \text{or} \qquad T < \frac{1}{2B}$$

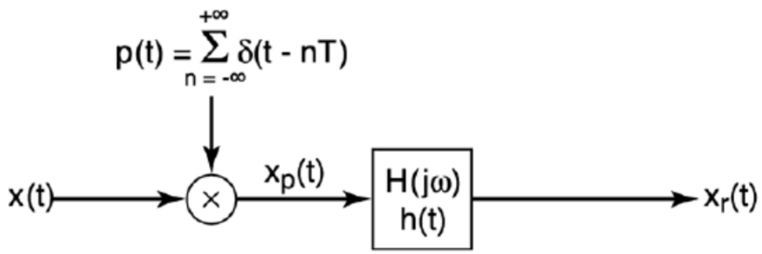
Signal Reconstruction in Practice

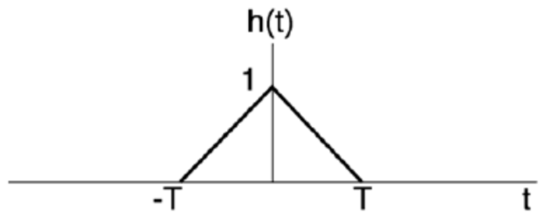
2. Zero-order hold



3. First-order hold (linear interpolation)



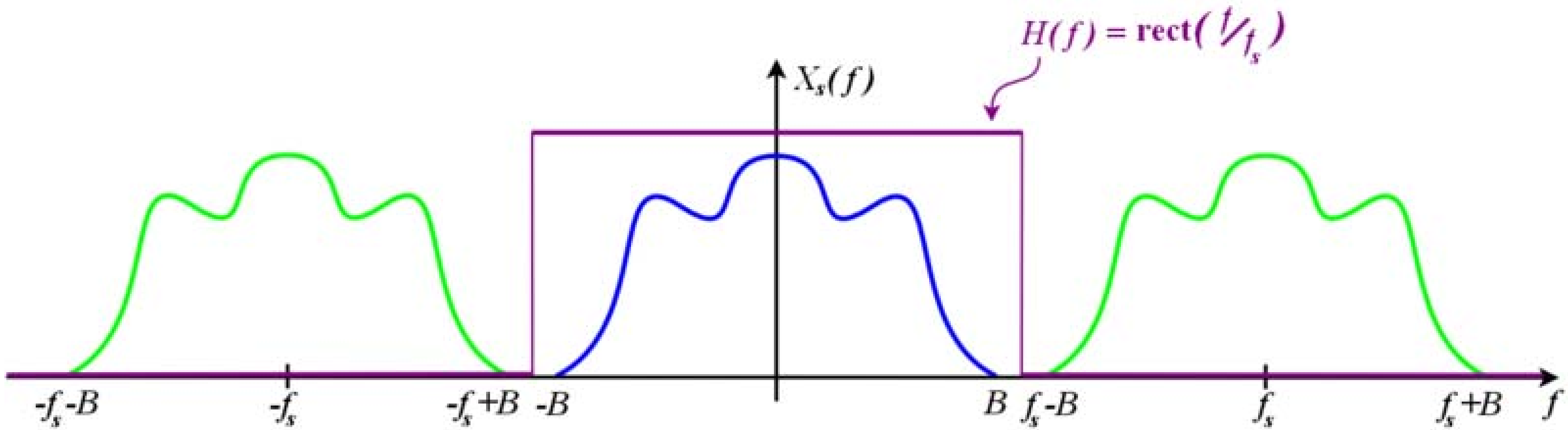


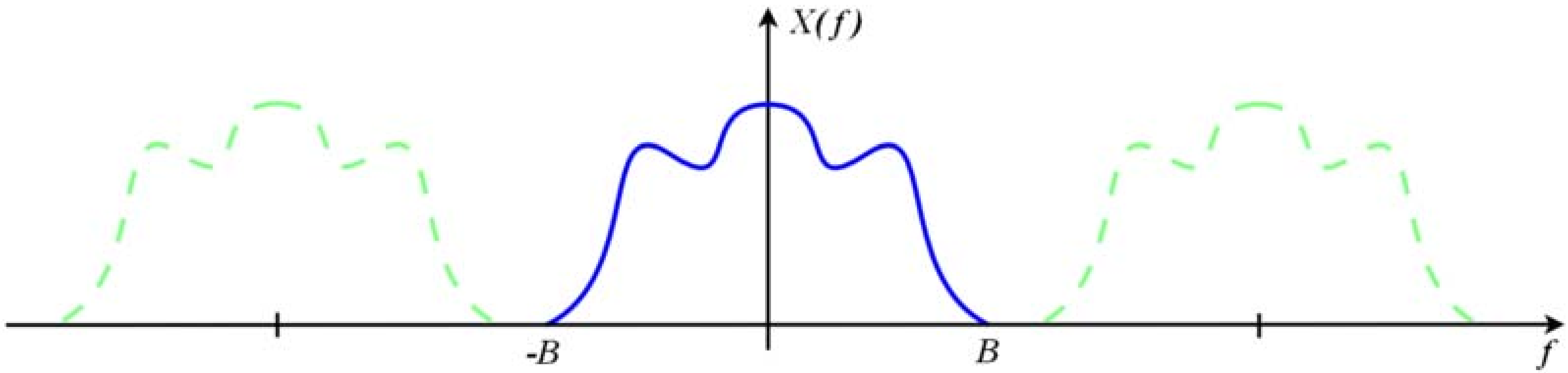


From Prof. A. S. Willsky, Signals and Systems course

Aliasing

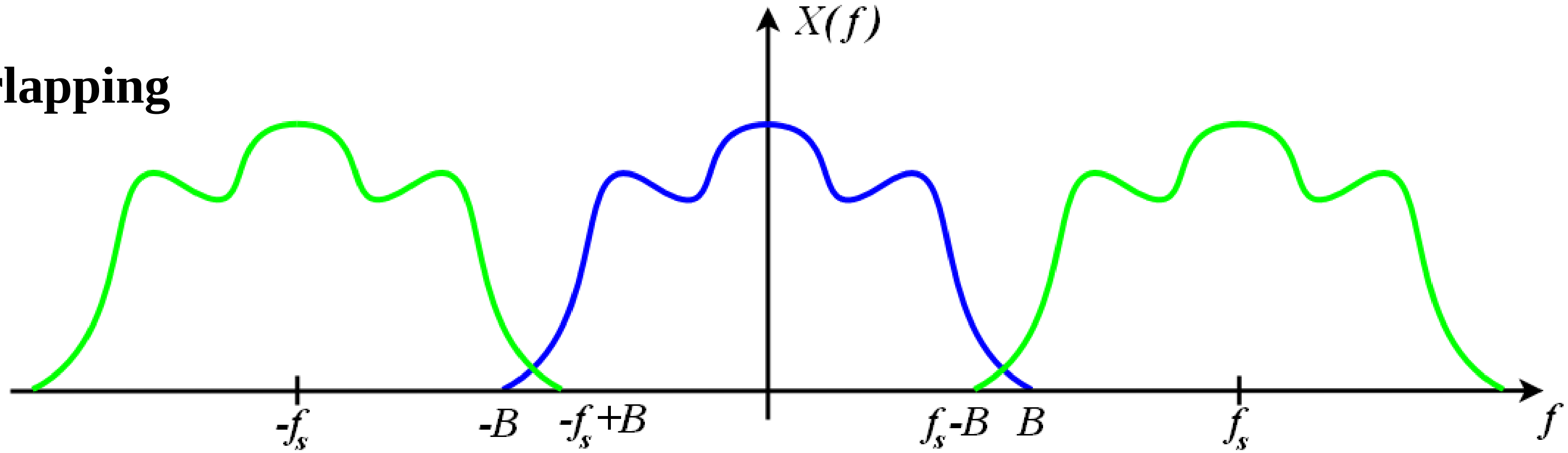
No Problems in Reconstruction



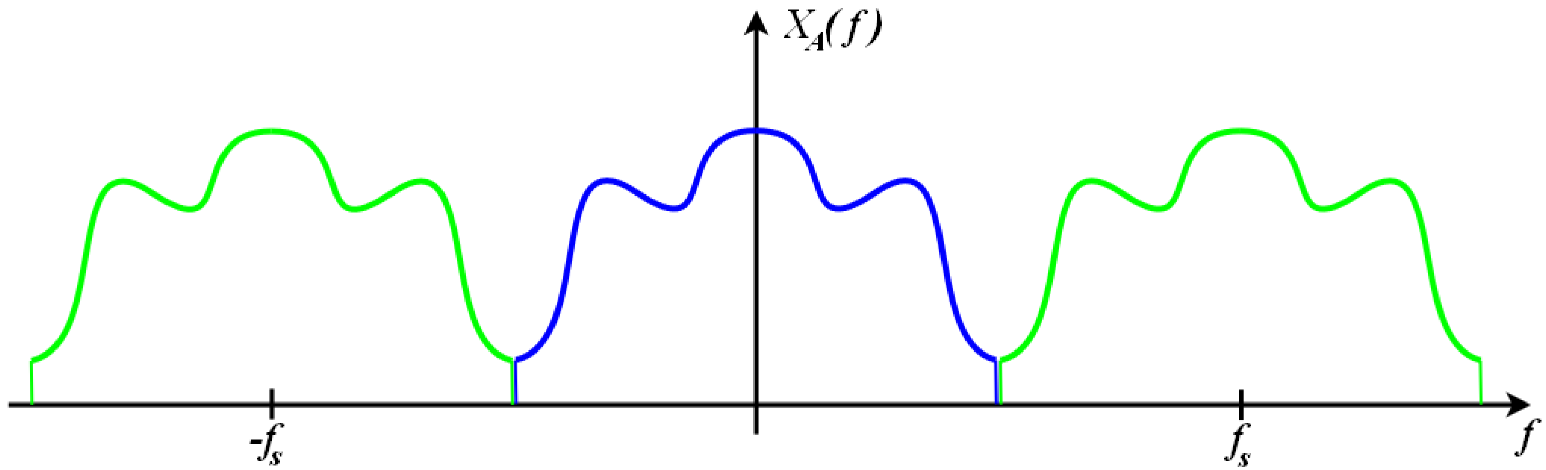


Reconstruction Problems

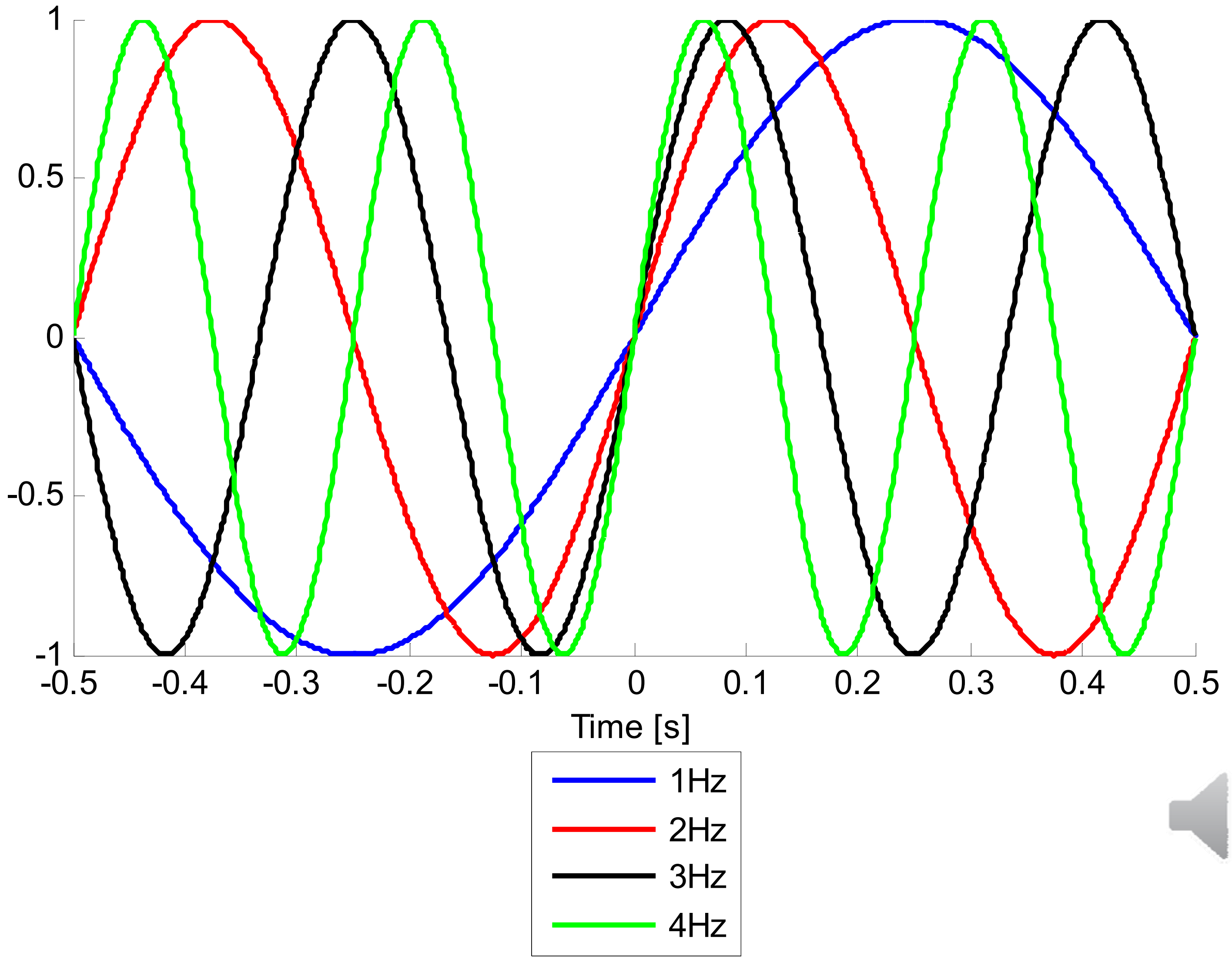
Overlapping



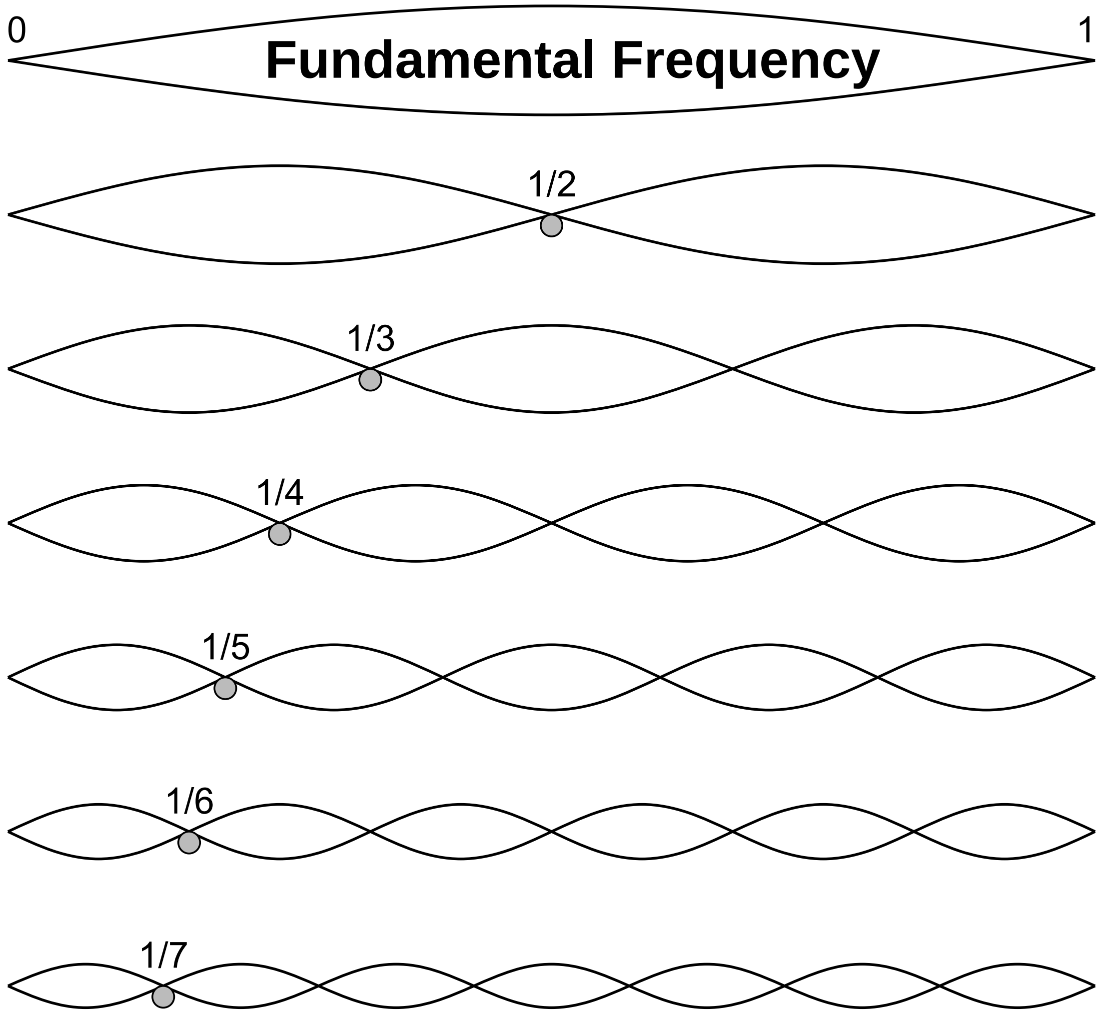
Alias



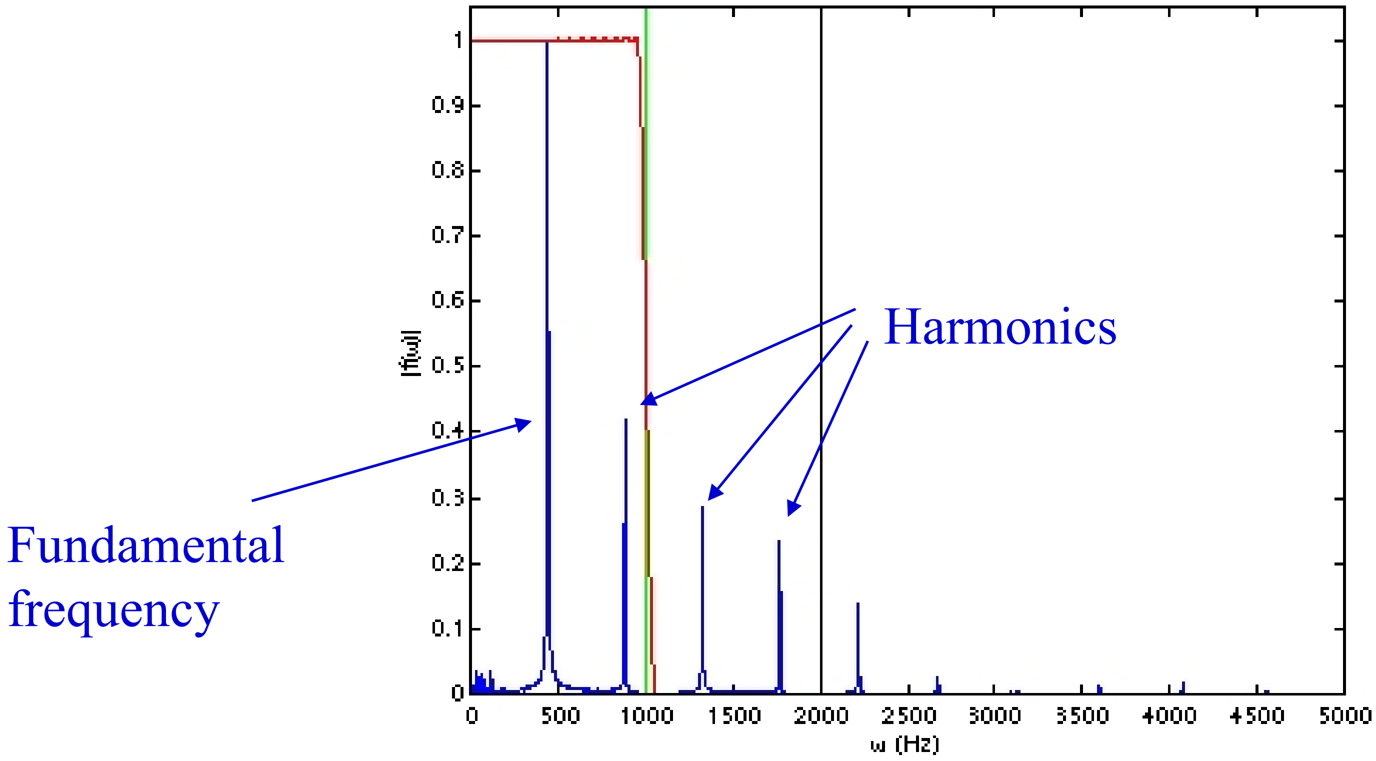
Harmonics



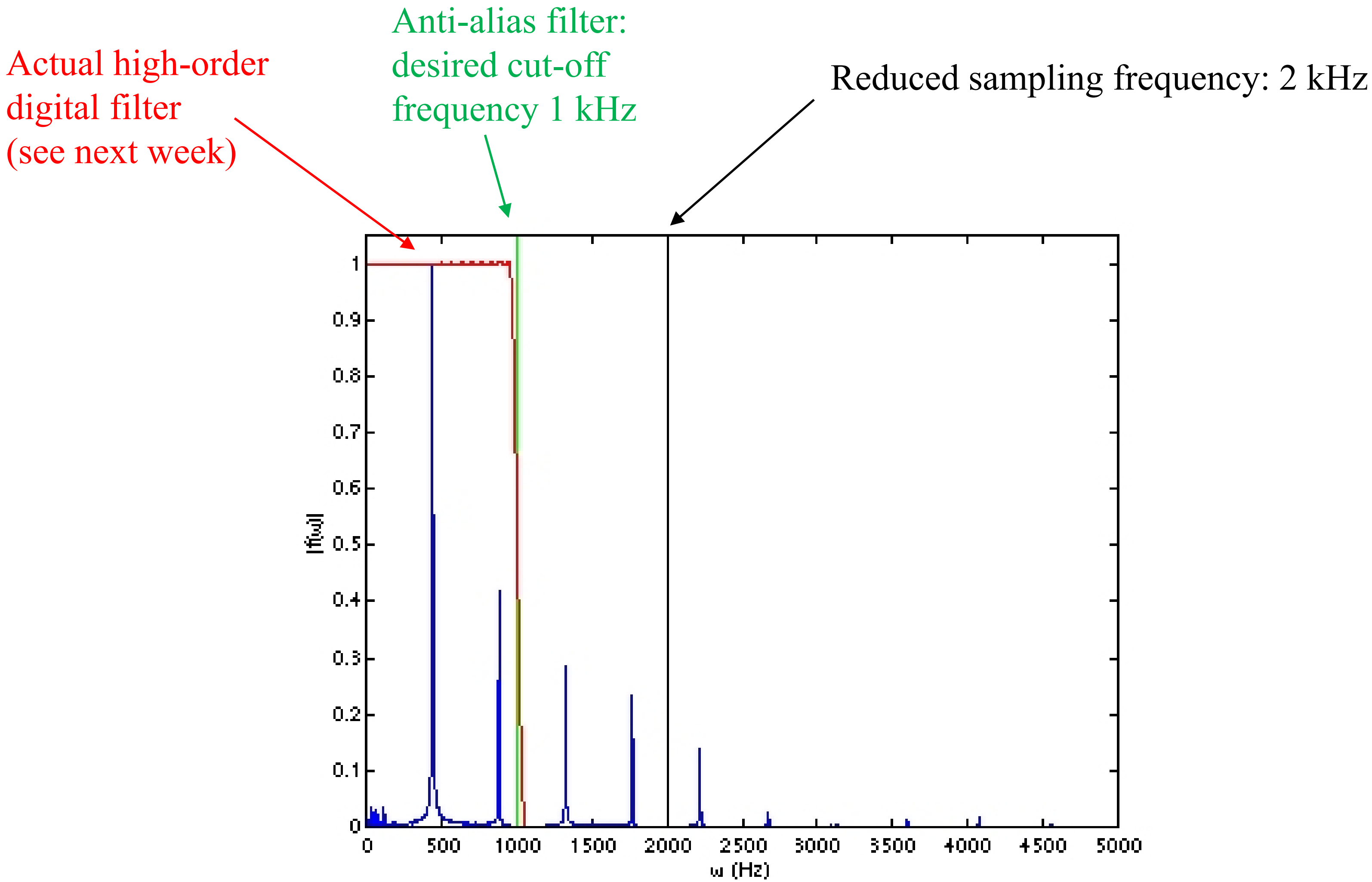
Harmonics



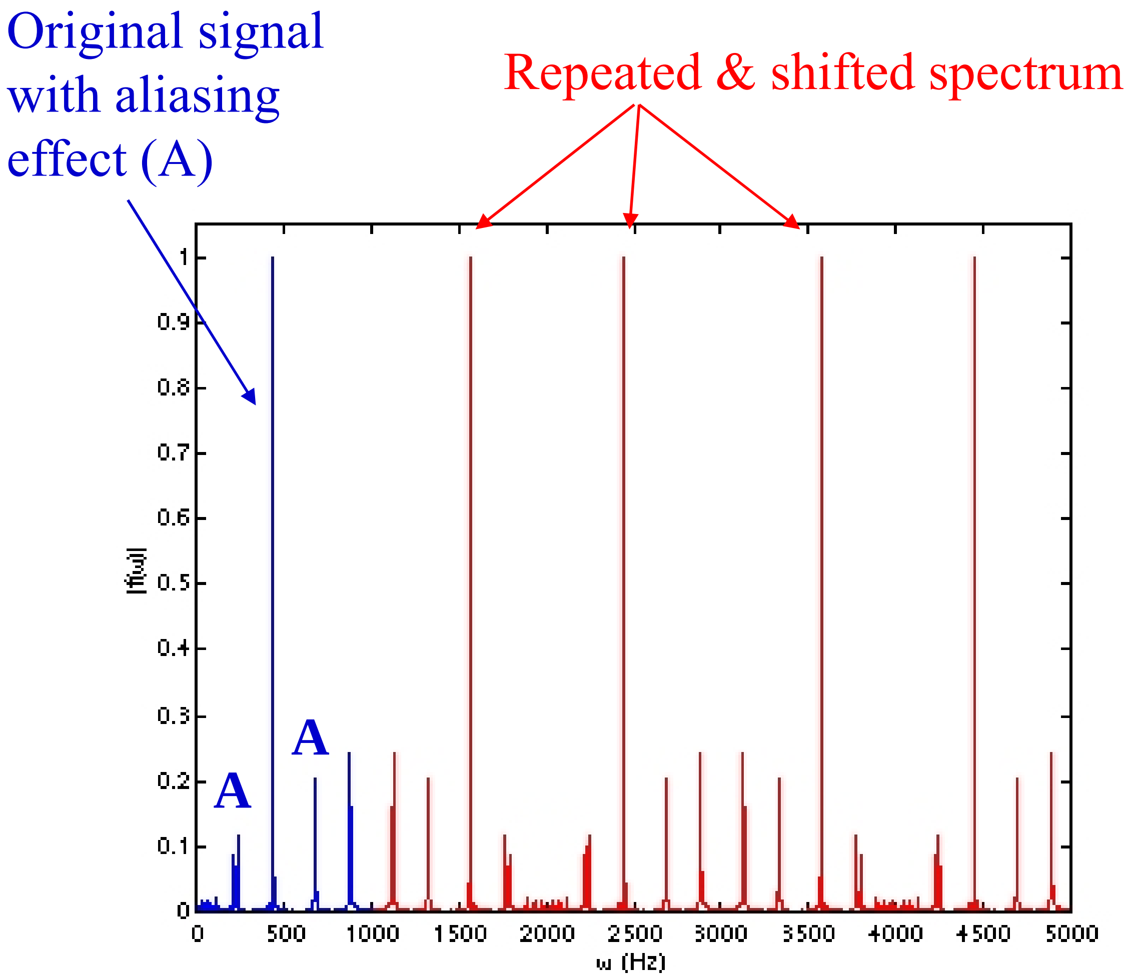
La-Tone (440 Hz) sampled at 44.1 kHz (CD standard)



La-Tone (440 Hz) sampled at 2 kHz



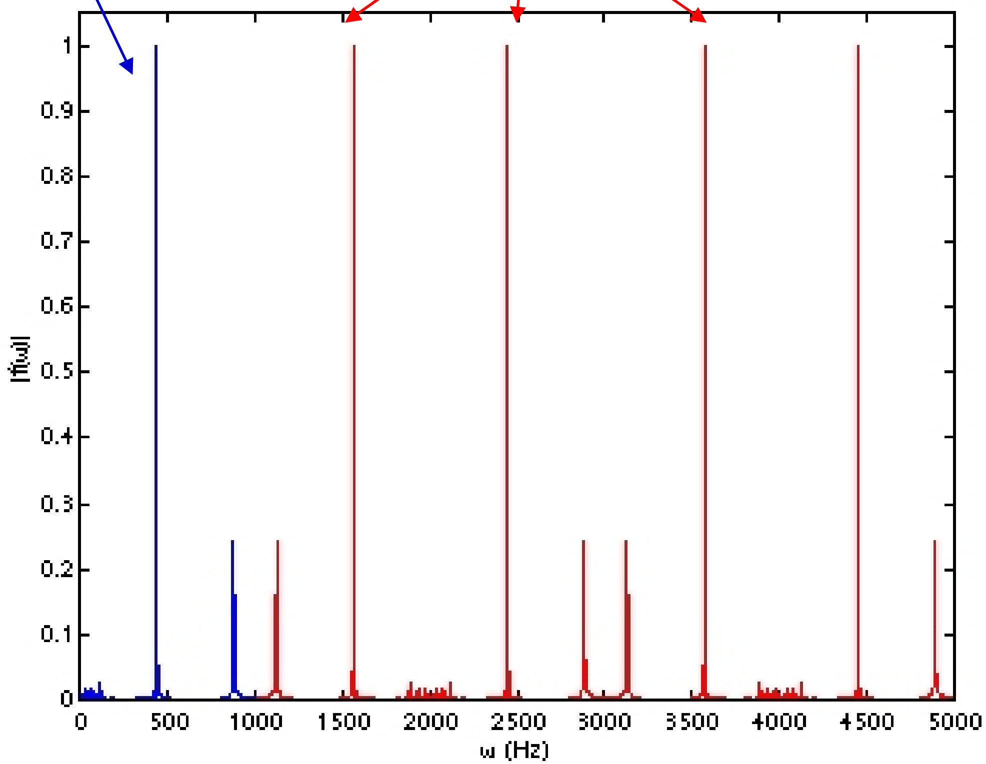
La-Tone (440 Hz) sampled at 2 kHz without filtering



La-Tone (440 Hz) sampled at 2 kHz filtered at 1 kHz

Original signal without aliases through anti-alias filtering

Repeated & shifted spectrum through sampling but without aliasing (anti-alias filter); will be eliminated through low-pass filtering at signal reconstruction stage



Aliasing Audio Examples

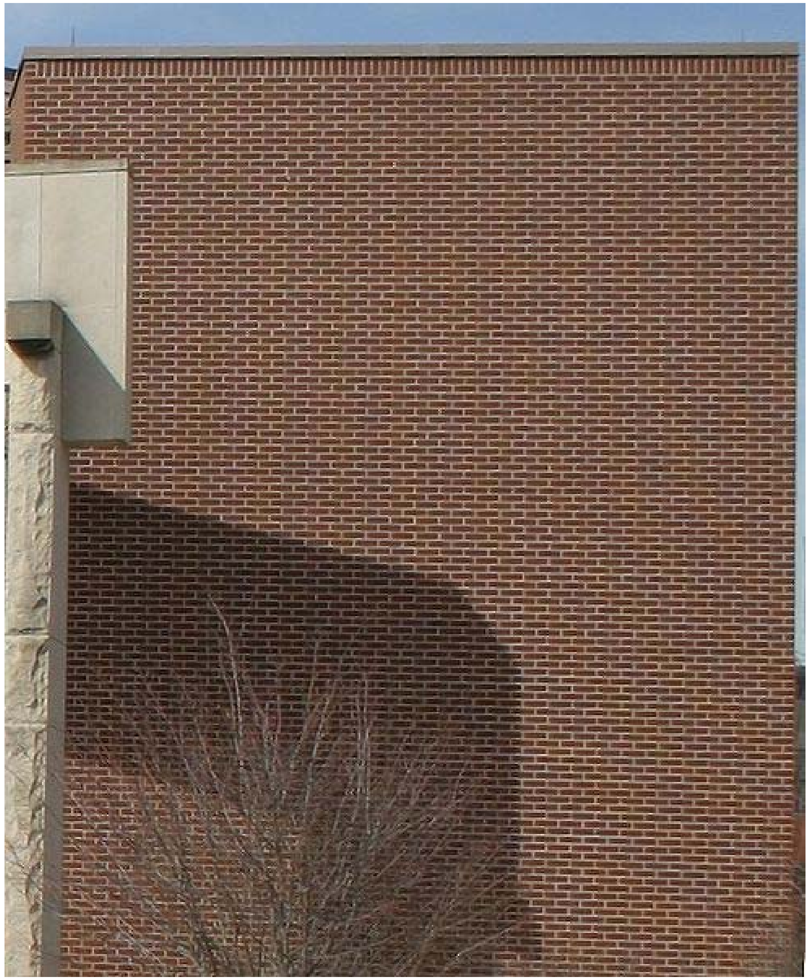
Original sound

Aliases 4 kHz

Correct sampling 4 kHz

Note: for instance cymbals (8-16 kHz) fully cut off, no matter what

Moiré Pattern



Aliasing Video Examples

https://www.youtube.com/watch?v=R-IVw8OKjvQ

http://www.youtube.com/watch?v=jHS9JGkEOmA

Conclusion

Take Home Messages

- Multiplication in time domain means convolution in frequency domain and vice versa
- Continuous vs. discrete convolution (analogy: FT vs. DFT)
- Sampling: multiplication of the signal with a periodic train of Dirac pulses and therefore repeated shifted spectrum in the frequency domain
- Aliasing: higher frequency “folded back” on original spectrum -> prevent proper signal reconstruction
- When you sample a signal …
  - Make sure you know what the maximum frequency f_{max} is or enforce it through an anti-alias low-pass filter
  - Make sure you sample at $f_s > 2 f_{max}$ (Nyquist rule)

Additional Literature – Week 5

- **Books:**
- J. H. McClellan, R. W. Schafer, M. A. Yoder "DSP First: A Multimedia Approach", Prentice Hall, 1999.
- A. Oppenheim and A. S. Willsky with S. Nawab, "Signals and Systems", Prentice Hall, 1997.