Signals, Instruments, and Systems – W6

Introduction to Signal Processing – Analysis and Synthesis of Filters
Outline

• Motivating examples
• Transforms, transfer functions, terminology
• Bode plots
• Examples of first order analog filters
• Digital filters
Acknowledgment for Selected Slides

Signals and Systems

Fall 2003
Lecture #1
Prof. Alan S. Willsky
4 September 2003

1) Administrative details
2) Signals
3) Systems
4) For examples ...

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Motivating Examples
Filtering Noisy Signals

day/night cycle

changing cloud cover

Solar radiation

low pass filter

high pass filter
Anti-Aliasing Filters

Reduced sampling frequency: 2 KHz

Anti-alias filter: desired cut-off frequency 1 KHz

Actual high-order digital filter

Example of Latone from Week 5 slides

Fundamental frequency

Harmonics
Filters for Signal Reconstruction

Graphic Illustration of Time-Domain Interpolation

Original CT signal

After sampling

After passing the LPF

From Prof. A. S. Willsky, Signals and Systems course
Relevant Transforms, Transfer Functions, Terminology
Transforms for Causal Systems

CAUSALITY

• A system is causal if the output does not anticipate future values of the input, i.e., if the output at any time depends only on values of the input up to that time.

• All real-time physical systems are causal, because time only moves forward. Effect occurs after cause. (Imagine if you own a noncausal system whose output depends on tomorrow’s stock price.)

• Causality does not apply to spatially varying signals. (We can move both left and right, up and down.)

• Causality does not apply to systems processing recorded signals, e.g. taped sports games vs. live broadcast.

From Prof. A. S. Willsky, Signals and Systems course
Motivation for More Transforms

Motivation for the Laplace Transform

- CT Fourier transform enables us to do a lot of things, e.g.
  - Analyze frequency response of LTI systems
  - Sampling
  - Modulation
- Why do we need yet another transform?
- One view of Laplace Transform is as an extension of the Fourier transform to allow analysis of broader class of signals and systems
- In particular, Fourier transform cannot handle large (and important) classes of signals and unstable systems, i.e. when

\[ \int_{-\infty}^{\infty} |x(t)| \, dt = \infty \]

From Prof. A. S. Willsky, Signals and Systems course
Laplace Transform

\[ F(s) = \mathcal{L}\{f(t)\} = \int_{0^-}^{\infty} e^{-st} f(t) \, dt \]

\[ s = \sigma + i\omega \]
The Fourier Transform is a special case of Laplace Transform

**Fourier:** frequency response (especially in signal processing)

**Laplace:** impulse response (especially in control)

\[ F(\omega) = F \{ f(t) \} = \mathcal{L} \{ f(t) \} \bigg|_{s = i\omega} \]

\[ = F(s) \bigg|_{s = i\omega} = \int_{-\infty}^{\infty} e^{-i\omega t} f(t) dt \]
Discrete-Time Fourier Transform

- Corresponds to the Fourier Transform for discrete-time signals (different from the Discrete Fourier Transform, a finite, bounded approximation of the Fourier Transform for digital devices)
- Transform discrete-time signals from time-domain to frequency domain

\[ X[\omega] = \sum_{n=-\infty}^{\infty} x[n] \cdot e^{-i\omega n} \]
Z Transform

- Corresponds to Laplace transform for time-discrete signals
- Transform signals from time-domain to frequency domain
- The Discrete-Time Fourier Transform is a special case of the Z Transform

\[ X(z) = \mathcal{Z} \{x[n]\} = \sum_{n=-\infty}^{\infty} x[n]z^{-n} \]

\[ z = Ae^{j\phi} \text{ or } z = A(\cos \phi + j \sin \phi) \]
Transform Overview

- Continuous signal
  - Time domain
  - Fourier/Laplace
  - Inverse Fourier/Laplace
  - DAC
  - ADC

- Discrete signal
  - Time domain
  - Z/DTFT
  - Inverse Z/DTFT

- Continuous signal
  - Frequency domain

- Discrete signal
  - Frequency domain
Filters

Analog

Circuit

Digital

Function

\[ y_1 \cdots y_n = f(x_1 \cdots x_n) \]
Transfer Functions of Filters

**Analog Circuit**

\[ V_R \quad I \quad V_C \]

\[ V_{in} \quad R \quad C \quad V_C \]

**Laplace Transf.**

\[ H(s) = \frac{v_c}{v_{in}} = \frac{1}{1 + RC_s} \]

**Numerator**

\[ \text{Denominator} \]

**Digital Function**

\[ y_1 \ldots y_n = f(x_1 \ldots x_n) \]

\[ H(z) = \frac{b_0 + b_1 z^{-1} + \ldots + b_N z^{-N}}{1 + a_1 z^{-1} + \ldots + b_M z^{-M}} \]

**Numerator**

\[ \text{Denominator} \]
Idealized Low Pass Filter

CT

$$H(j\omega)$$

$\omega_c$ — cutoff frequency

DT

$$H(e^{j\omega})$$

Note: $|H| = 1$ and $\angle H = 0$ for the ideal filters in the passbands, no need for the phase plot.
Low Pass Filter

CT
$|H(j\omega)|$

$\omega$

Stopband
Passband
Stopband

Note for DT:
$H(e^{j\omega}) = H(e^{j(\omega+2\pi)})$

DT
$|H(e^{j\omega})|$

low frequency

low frequency

$\omega$

-2$\pi$
$-\pi$
$\pi$
$2\pi$
Idealized High Pass Filter

\[ H(j\omega) \]

\[ H(e^{j\omega}) \]

\[ \omega_c, \pi, 2\pi - \omega_c, 2\pi \]
High Pass Filter

Remember:

\((-1)^n = e^{j\pi n}\)
\(\pi = \text{highest frequency in DT}\)

\(H(e^{j\omega})\)

High frequency

High frequency
Idealized Band Pass Filter

CT

DT
Band Pass Filter

CT

$|H(j\omega)|$

$-\omega_0$ $\omega_0$

DT

$|H(e^{j\omega})|$

$-\pi$ $-\omega_0$ $\omega_0$ $\pi$
Bode Plots
Bode plot: The Example of a Low Pass Filter

not to scale!
## Decibel

\[
G_{dB} = 20\log_{10}\left(\frac{V_2}{V_1}\right)
\]

\(V_2 > V_1 \rightarrow G_{dB} > 1\) (gain)

\(V_2 < V_1 \rightarrow G_{dB} < 1\) (damping)

<table>
<thead>
<tr>
<th>Source of sound</th>
<th>Sound pressure</th>
<th>Sound pressure level</th>
</tr>
</thead>
<tbody>
<tr>
<td>Jet engine at 30 m</td>
<td>630 Pa</td>
<td>150 dB</td>
</tr>
<tr>
<td>Rifle being fired at 1 m</td>
<td>200 Pa</td>
<td>140 dB</td>
</tr>
<tr>
<td>Threshold of pain</td>
<td>100 Pa</td>
<td>130 dB</td>
</tr>
<tr>
<td>Hearing damage (due to short-term exposure)</td>
<td>20 Pa</td>
<td>approx. 120 dB</td>
</tr>
<tr>
<td>Jet at 100 m</td>
<td>6 – 200 Pa</td>
<td>110 – 140 dB</td>
</tr>
<tr>
<td>Jack hammer at 1 m</td>
<td>2 Pa</td>
<td>approx. 100 dB</td>
</tr>
<tr>
<td>Hearing damage (due to long-term exposure)</td>
<td>(6 \times 10^{-1}) Pa</td>
<td>approx. 85 dB</td>
</tr>
<tr>
<td>Major road at 10 m</td>
<td>(2 \times 10^{-1} – 6 \times 10^{-1}) Pa</td>
<td>80 – 90 dB</td>
</tr>
<tr>
<td>Passenger car at 10 m</td>
<td>(2 \times 10^{-2} – 2 \times 10^{-1}) Pa</td>
<td>60 – 80 dB</td>
</tr>
<tr>
<td>TV (set at home level) at 1 m</td>
<td>(2 \times 10^{-2}) Pa</td>
<td>approx. 60 dB</td>
</tr>
<tr>
<td>Normal talking at 1 m</td>
<td>(2 \times 10^{-3} – 2 \times 10^{-2}) Pa</td>
<td>40 – 60 dB</td>
</tr>
<tr>
<td>Very calm room</td>
<td>(2 \times 10^{-4} – 6 \times 10^{-4}) Pa</td>
<td>20 – 30 dB</td>
</tr>
<tr>
<td>Leaves rustling, calm breathing</td>
<td>(6 \times 10^{-5}) Pa</td>
<td>10 dB</td>
</tr>
<tr>
<td>Auditory threshold at 1 kHz</td>
<td>(2 \times 10^{-5}) Pa</td>
<td>0 dB</td>
</tr>
</tbody>
</table>
Bode Plot - Rules

• Zero (numerator = 0)
  – Amplitude: 20 dB/decade
  – Phase: $90^\circ$, $45^\circ$ /decade, starting 1 decade before zero

• Pole (denominator = 0)
  – Amplitude: -20 dB/decade
  – Phase: $-90^\circ$, $-45^\circ$ /decade, starting 1 decade before pole
Bode plot (magnitude)

Zero (numerator = 0) Amplitude: 20 dB/decade
Pole (denominator = 0) Amplitude: -20 dB/decade
Bode plot (phase)

Zero (numerator = 0): \(90^\circ, 45^\circ\) /decade, starting 1 decade before zero
Pole (denominator = 0): \(-90^\circ, -45^\circ\) /decade, starting 1 decade before pole
Examples of First Order Analog Filters
Low Pass Filter - RC circuit

\[ |V_{\text{out}}| = |V_{\text{in}}| \frac{1}{\sqrt{1 + \omega^2 R^2 C^2}} \]

Breakpoint
\[ \omega_B = \frac{1}{RC} \]

\[ \frac{|V_{\text{out}}|}{|V_{\text{in}}|} \text{ in decibels} \]
Bode plot –
First order Low Pass Filter
High Pass Filter – RC circuit

\[ |V_{out}| = |V_{in}| \frac{\omega RC}{\sqrt{1 + \omega^2 R^2 C^2}} \]

Breakpoint

\[ \omega_B = \frac{1}{RC} \]

\[ \frac{|V_{out}|}{|V_{in}|} \text{ in decibels} \]
Bode plot – first order high pass filter
Digital Filters
The General Case

Difference equation:

\[ y[n] = - \sum_{k=1}^{N} a_k y[n - k] + \sum_{k=0}^{M} b_k x[n - k] \]

\textbf{Z-transform:}

\[ Y(z) + \sum_{k=1}^{N} a_k Y(z) z^{-k} = \sum_{k=0}^{M} b_k X(z) z^{-k} \]

\textbf{Transfer function:}

\[ H[z] = \frac{Y[z]}{X[z]} = \frac{\sum_{k=1}^{M} b_k z^{-k}}{1 + \sum_{k=0}^{N} a_k z^{-k}} \]
An Example

Difference equation:

\[ y[n] = x[n] + 2x[n - 1] + x[n - 2] - \frac{1}{4} y[n - 1] + \frac{3}{8} y[n - 2] \]

Transfer function:

\[
H[z] = \frac{Y[z]}{X[z]} = \frac{1 + 2z^{-1} + z^{-2}}{1 + \frac{1}{4}z^{-1} - \frac{3}{8}z^{-2}} = \frac{z^2 + 2z + 1}{z^2 + \frac{1}{4}z - \frac{3}{8}} = \frac{(z + 1)^2}{(z - \frac{1}{2})(z + \frac{3}{4})}
\]
Nonrecursive Discrete-Time Filters

\[ y[n] = \sum_{k=-N}^{M} b_k x[n - k] \]

- Finite Impulse Response (FIR)
- Unrelated to continuous time filtering
- If \( b_k \neq 0 \) for any \( n < 0 \) then noncausal
Nonrecursive Discrete-Time Filters 3

point moving averaging

- $y[n] = \frac{1}{3} (x[n - 1] + x[n] + x[n + 1])$
- $h[n] = \frac{1}{3} (\delta[n - 1] + \delta[n] + \delta[n + 1])$ (FIR)
Nonrecursive Discrete-Time Filters

3 point moving averaging

- \( y[n] = \frac{1}{3} (x[n - 1] + x[n] + x[n + 1]) \)
- \( h[n] = \frac{1}{3} (\delta[n - 1] + \delta[n] + \delta[n + 1]) \) (FIR)
- \( H(e^{j\omega}) = \frac{1}{3} (e^{j\omega} + 1 + e^{-j\omega}) = \frac{1}{3} (1 + 2 \cos(\omega)) \)

\[ |H(e^{j\omega})| \]

Low pass filter
Nonrecursive Discrete-Time Filters

Simple Highpass Filter

- $y[n] = \frac{1}{2} (x[n] - x[n - 1])$
- $h[n] = \frac{1}{2} (\delta[n] - \delta[n - 1])$ (FIR)
Nonrecursive Discrete-Time Filters
Simple Highpass Filter

- \( y[n] = \frac{1}{2} (x[n] - x[n - 1]) \)
- \( h[n] = \frac{1}{2} (\delta[n] - \delta[n - 1]) \) \((FIR)\)
- \( H(e^{j\omega}) = \frac{1}{2} (1 - e^{-j\omega}) = je^{j\omega} \sin(\omega/2) \)

\[ |H(e^{j\omega})| \]

High pass filter
Order of Filters
Filter Order and Type

• Several filters exists and are defined by the polynomials at the numerator/denominator (Finite Impulse Response, Bessel, Butterworth, Tschebishev, etc.)

• 1st order is equivalent to 20dB per decade

• Each successive order adds 20dB per decade

• Filter with a high order are closer to the ideal filter (rectangular function)
Filter Order

Analog

Filter order: 3
++ faster cutoff
-- more components
-- higher power consumption

Digital

\[ y[n] = a_0 x[n] \]
\[ y[n] = a_0 x[n] + a_1 x[n - 1] \]
\[ y[n] = a_0 x[n] + a_1 x[n - 1] + a_2 x[n - 2] \]
\vdots

Filter order: 1
++ faster cutoff
-- more computation

Filter order: 2
++ faster cutoff
-- higher power consumption

\vdots
Conclusion
Take-Home Messages

• Filters allow a number of operations (e.g., noise removal, contrast enhancement, anti-aliasing, etc.)
• Their transfer function can be represented in time and frequency domain
• They are often easier to design in the frequency domain
• Bode plots allows for analysis of filter response
• Filters are characterized by different order and coefficient distributions
• Programmable digital components (e.g. microcontrollers, DSPs) allow for easy encoding of digital filters
Additional Reading

Books

• Ronald W. Schafer and James H. McClellan

• A. Oppenheim and A. S. Willsky with S. Hamid,