Signals, Instruments, and Systems – W10

An Introduction to Mobile Robotics – Localization via Proprioceptive and Exteroceptive Sensing

Outline

- Augmenting odometry with feature-based localization in 1D
- The Kalman Filter algorithm in 1D
- The Kalman Filter algorithm in multiple dimensions

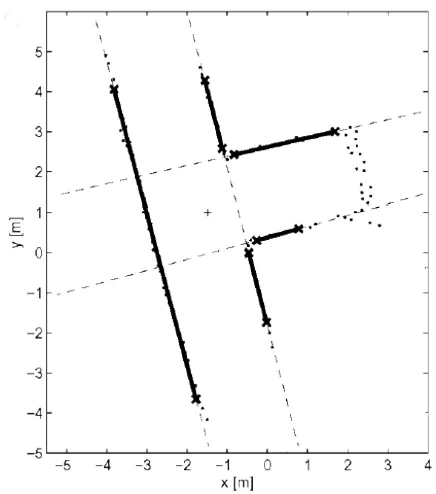
1D Localization using Odometry and Environmental Features

Features

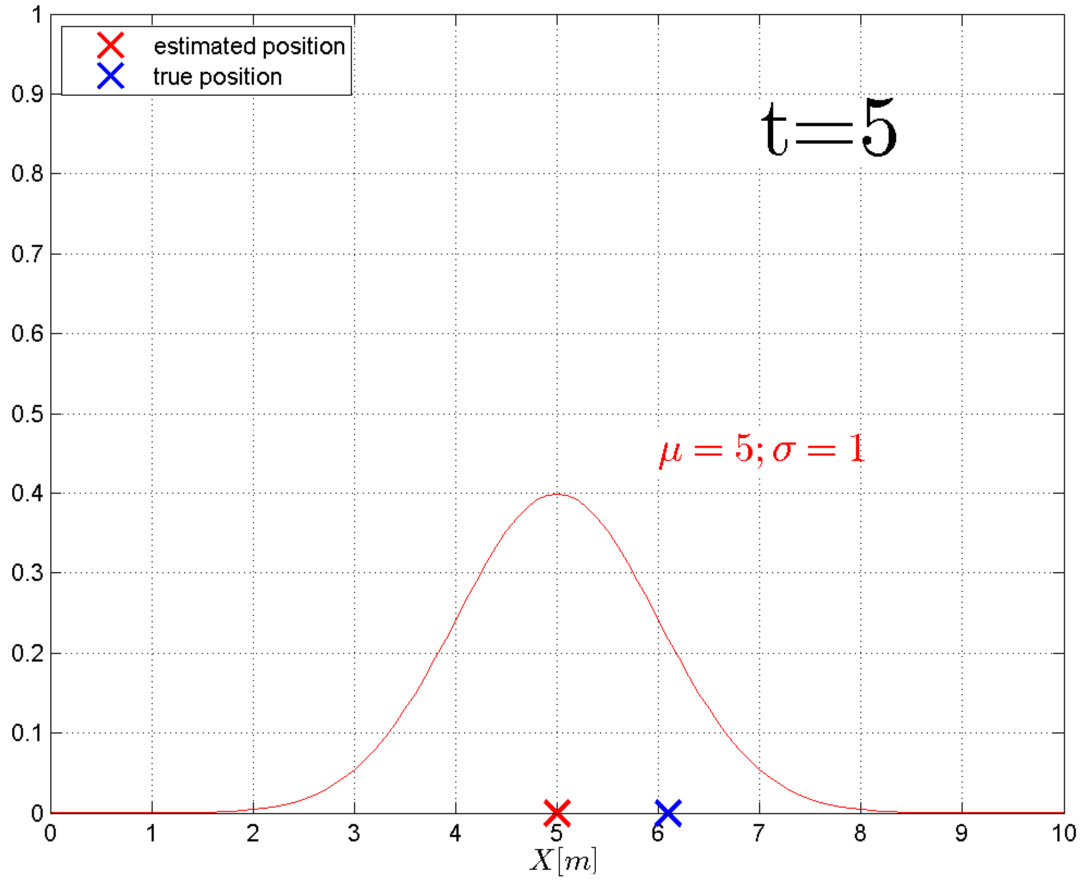
- Odometry based position error grows without bound.
- Use relative measurement to features (“landmarks”) to reduce position uncertainty
- ***Feature***:
  - **Uniquely** identifiable
  - Position is **known**
  - We can obtain relative measurements between robot and feature (usually angle or range).
- Examples:
  - Doors, walls, corners, hand rails
  - Buildings, trees, lanes
  - GNSS satellites

Automatic Feature Extraction

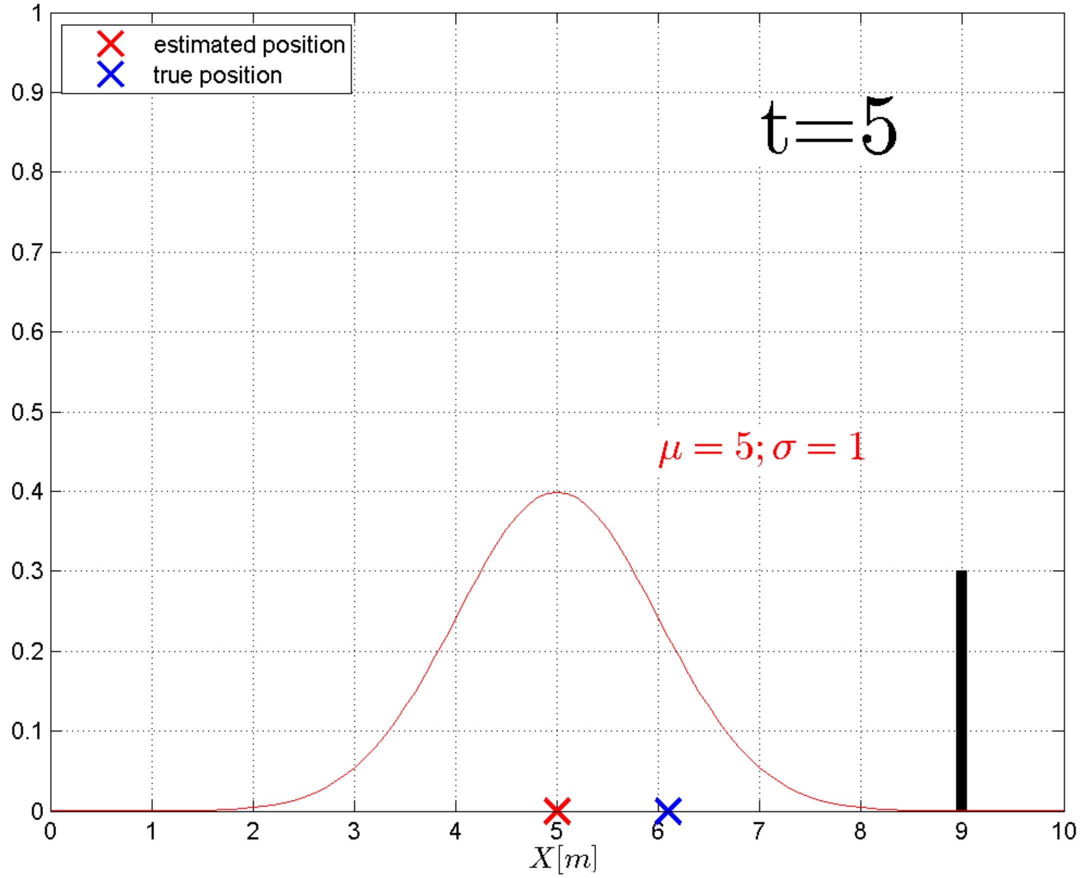
- High-level features:
  - Doors, persons
- Simple visual features:
  - Edges (Canny Edge Detector 1983)
  - Corner (Harris Corner Detector 1988)
  - ***S***cale ***I***nvariant ***F***eature ***T***ransformation (2004)
- Simple geometric features
  - Lines
  - Corners
- “Binary” feature



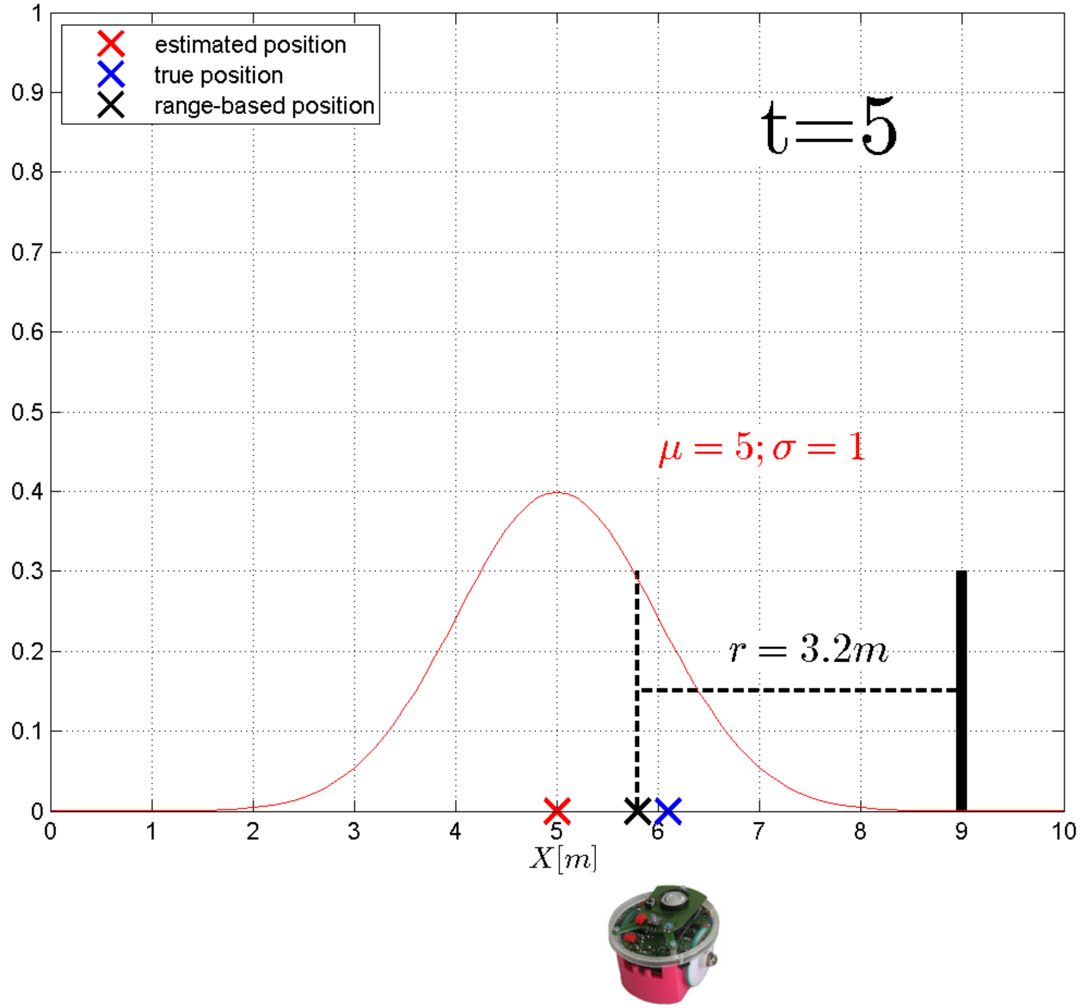
Feature-Based Localization



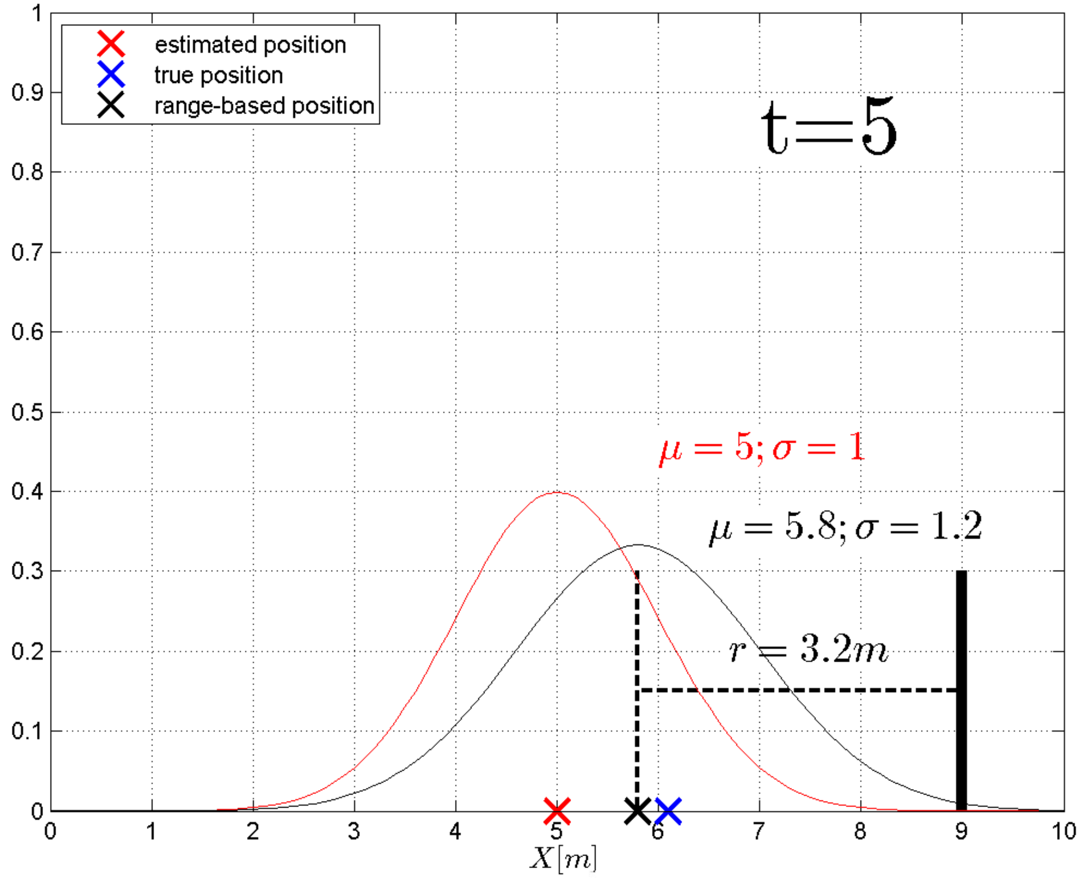
Feature-Based Localization



Feature-Based Localization



Feature-Based Localization

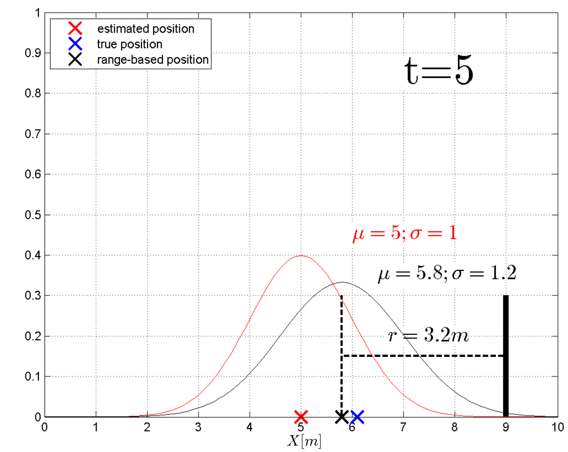


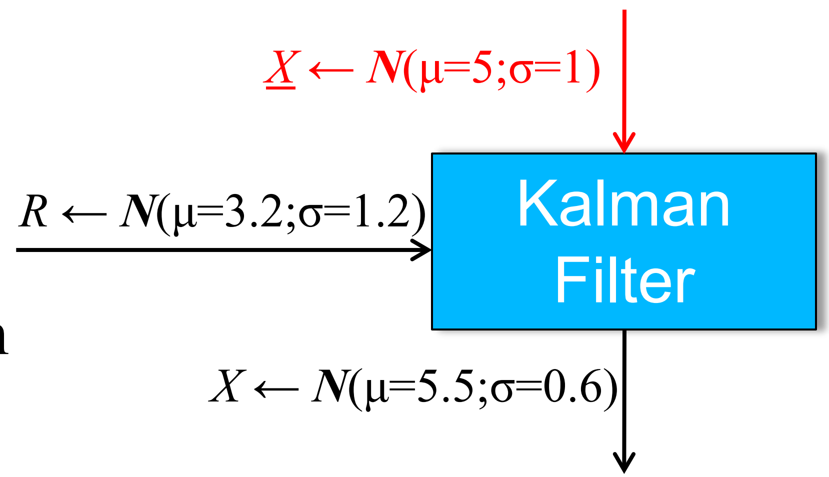
Sensor Fusion

- Given:
  - Position estimate $\underline{X} \leftarrow N(\mu=5;\sigma=1)$
  - Range estimate $R \leftarrow N(\mu=3.2;\sigma=1.2)$

    What is the best estimate AFTER incorporating R ?

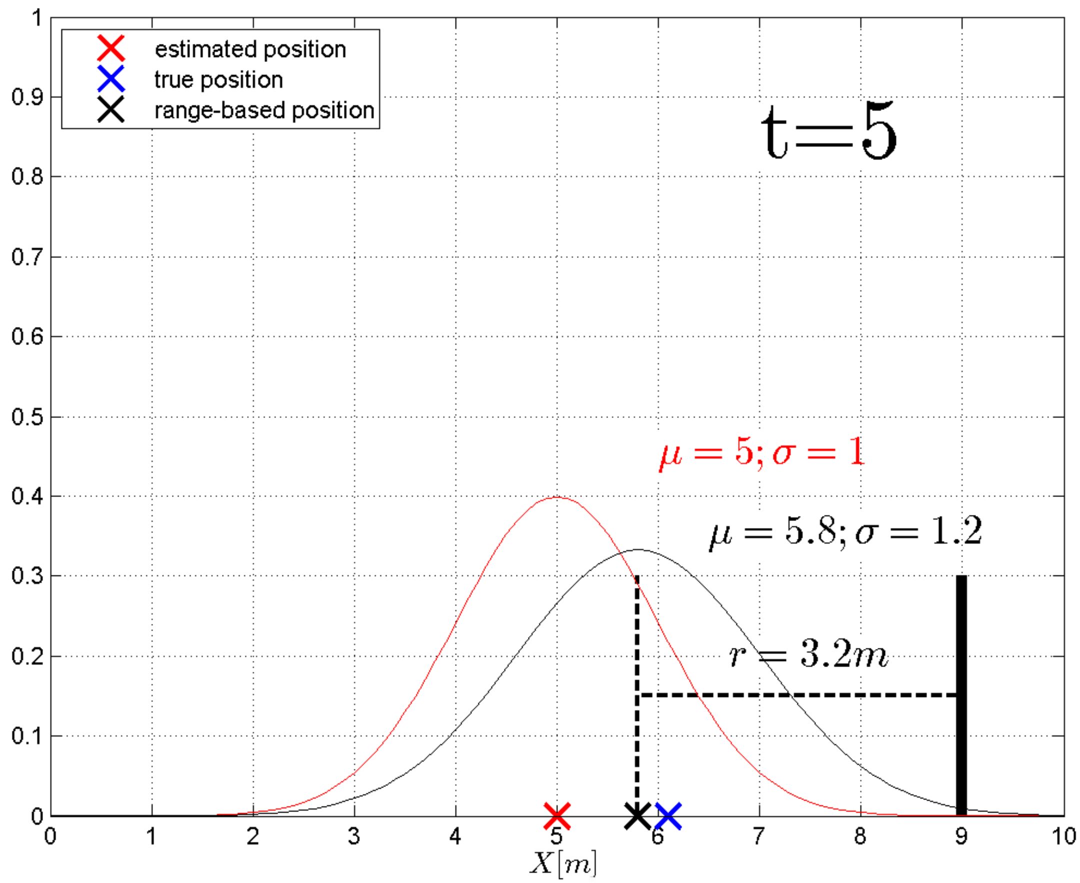
$\rightarrow$ ***Kalman Filter***



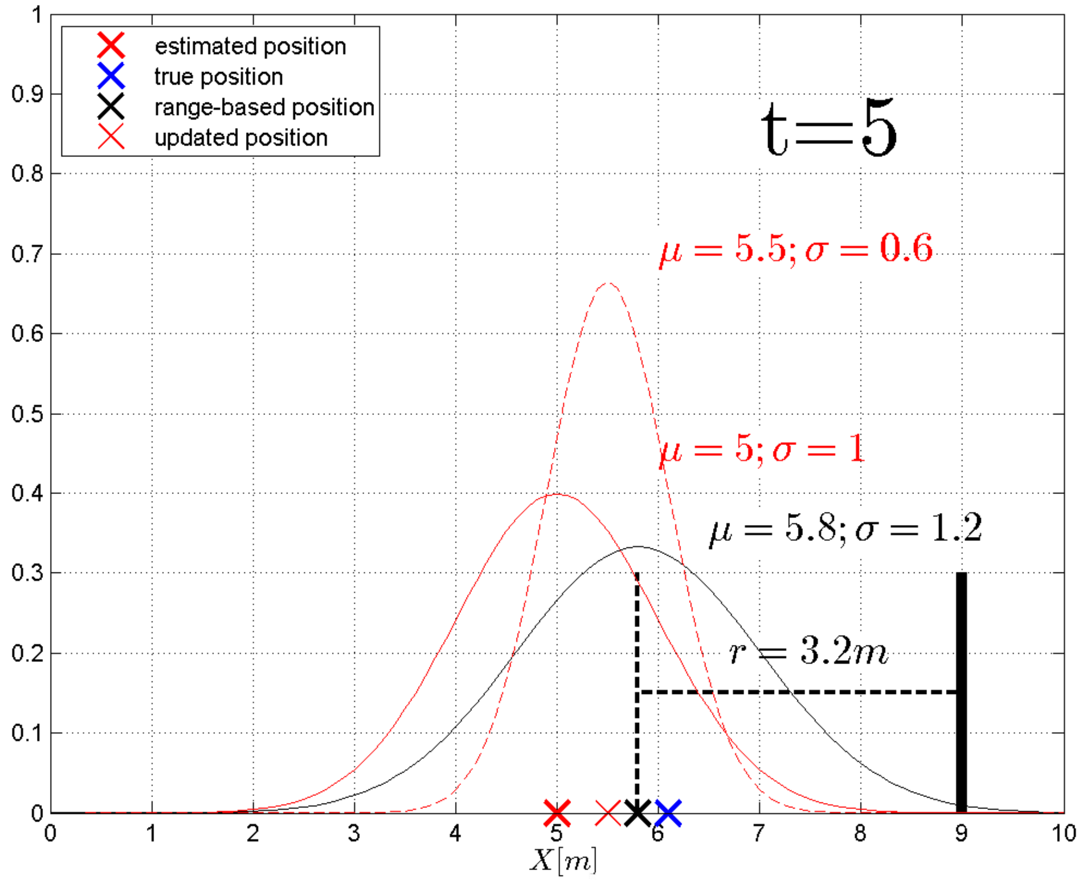


- Requires:
  - White Gaussian noise distribution for all measurements
  - Linear motion and measurement model
  - Position of the measured features

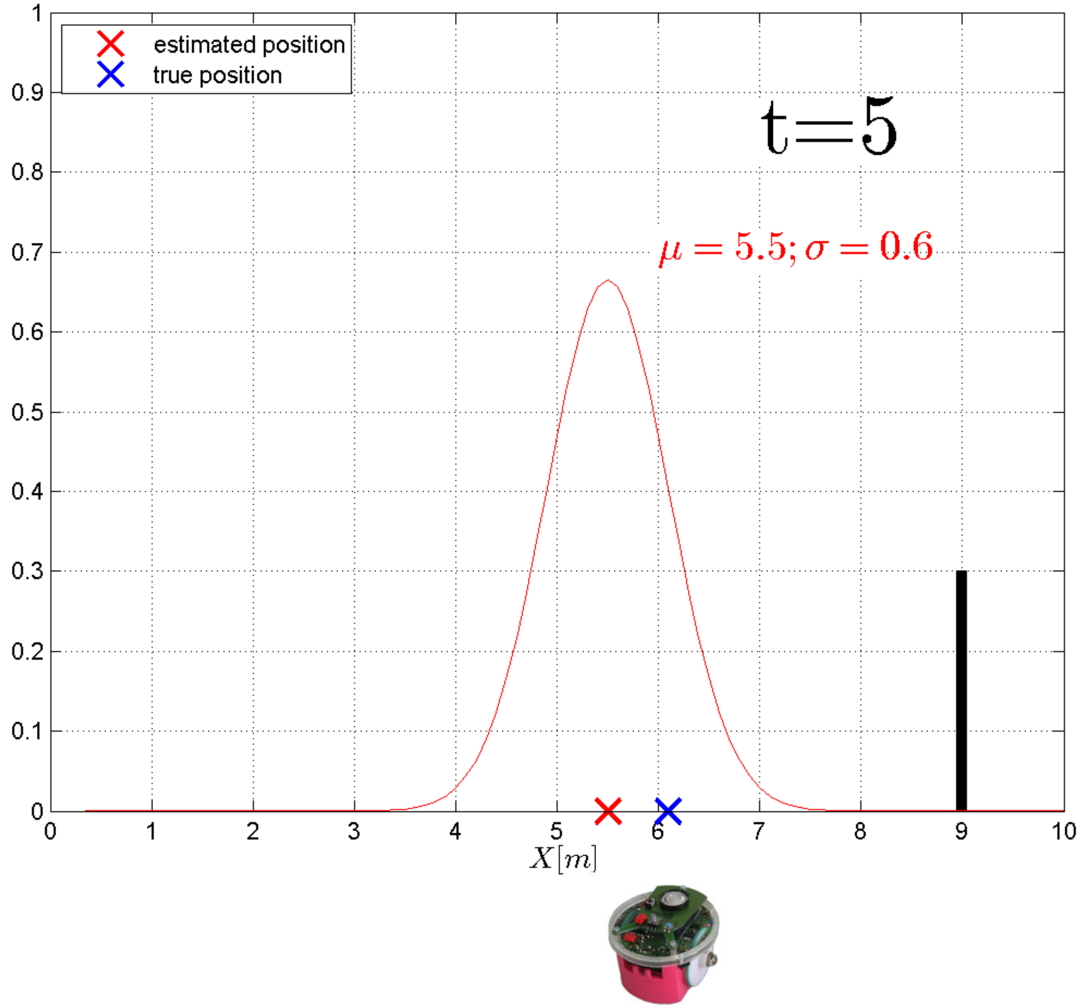
Feature-Based Localization



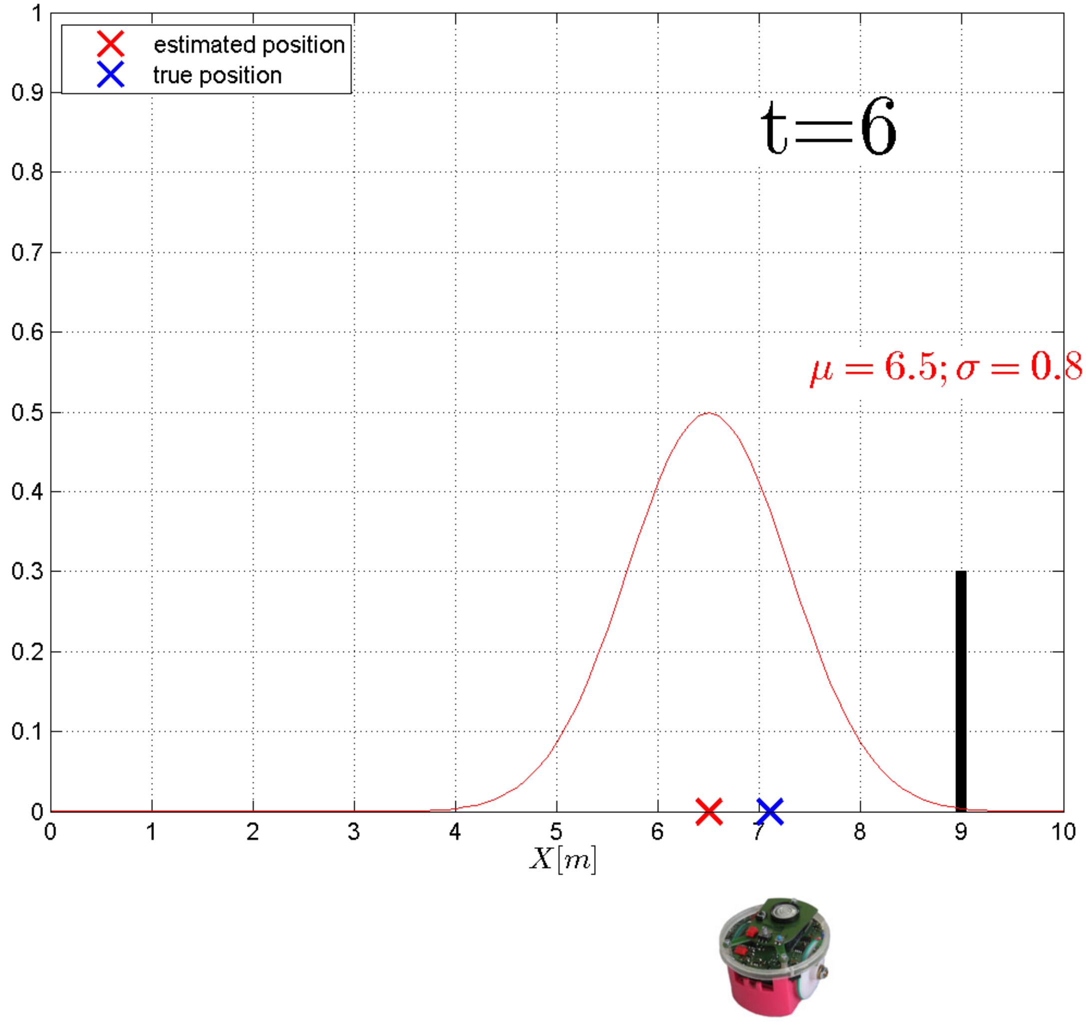
Feature-Based Localization



Feature-Based Localization



Feature-Based Localization



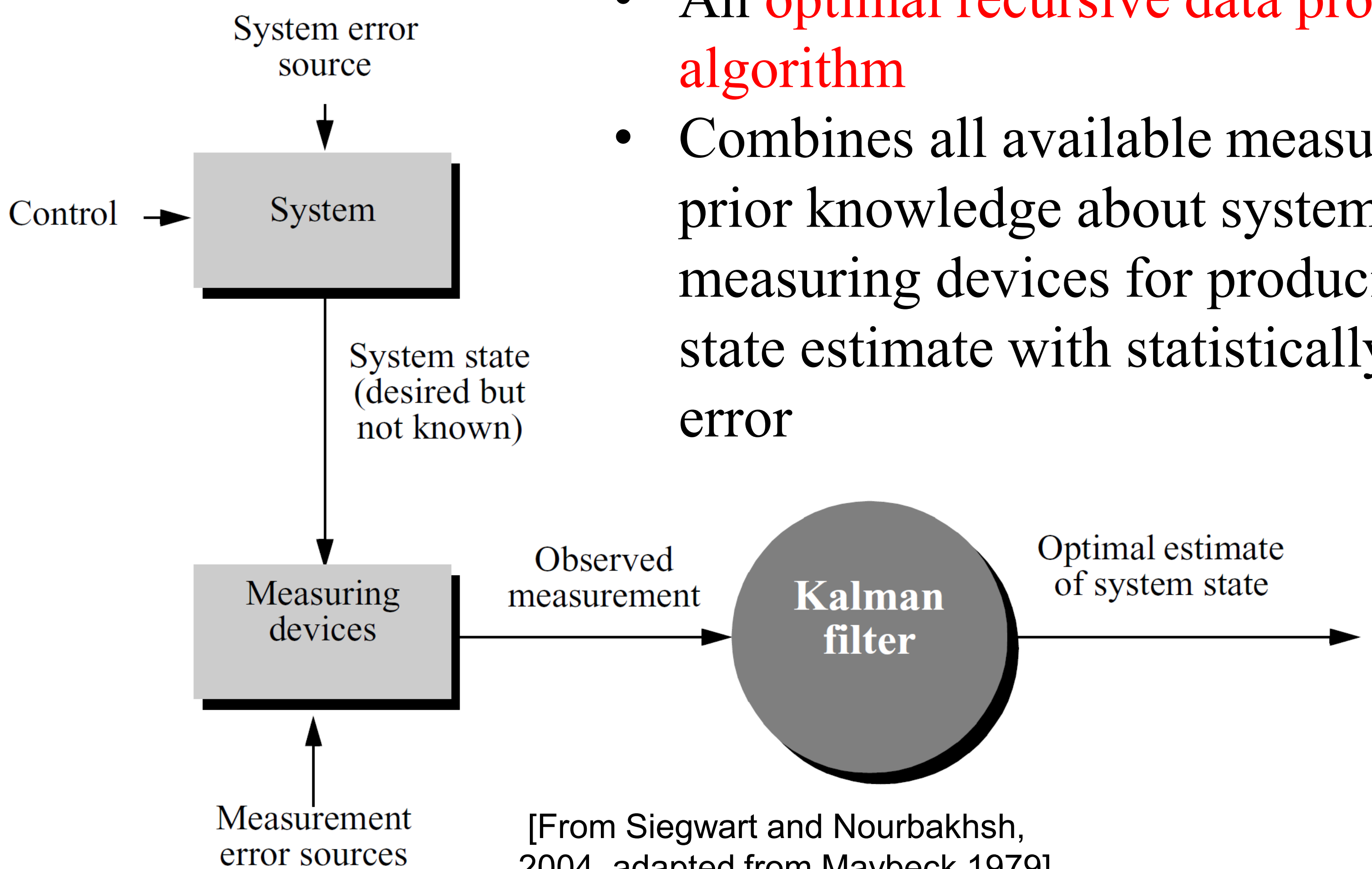
The Kalman Filter Algorithm for 1D Problems

Motivation for Stochastic Models and Estimation Methods

- A (mathematical) system model is necessarily an approximation of the real system: system structure is often appropriate but parameters are affected by uncertainties
- Non-deterministic sources of noise: we can neither control stochastic disturbances, nor get rid of stochastic sensor noise by calibration, nor model stochastic sources deterministically
- Sensors are noisy and can only partially observe the system state in all its details
- The applicability of estimation methods goes way beyond localization problems

Kalman Filter - Overview

- An optimal recursive data processing algorithm
- Combines all available measurement data, prior knowledge about system and measuring devices for producing a system state estimate with statistically minimized error



[From Siegwart and Nourbakhsh, 2004, adapted from Maybeck 1979]

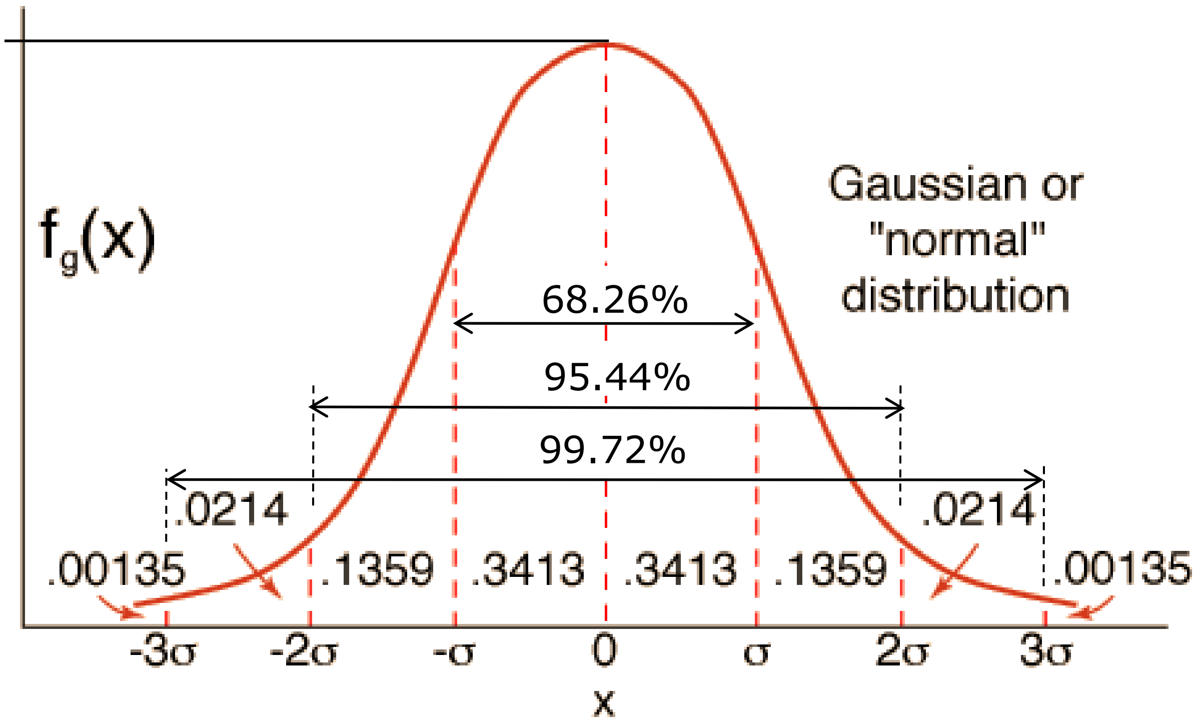
Kalman Filter - Assumptions

- Linear system and measurement model
- White Gaussian system and measurement noise

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) \equiv \mathcal{N}(\mu, \sigma^2)$$

Notes on noise:

- White: uncorrelated in time (used to reproduce a generalized stochastic process in time-discrete systems)
- Gaussian: probability density of amplitude follows bell-shaped curve



x = μ

A Simple 1D Positioning Example

t = time

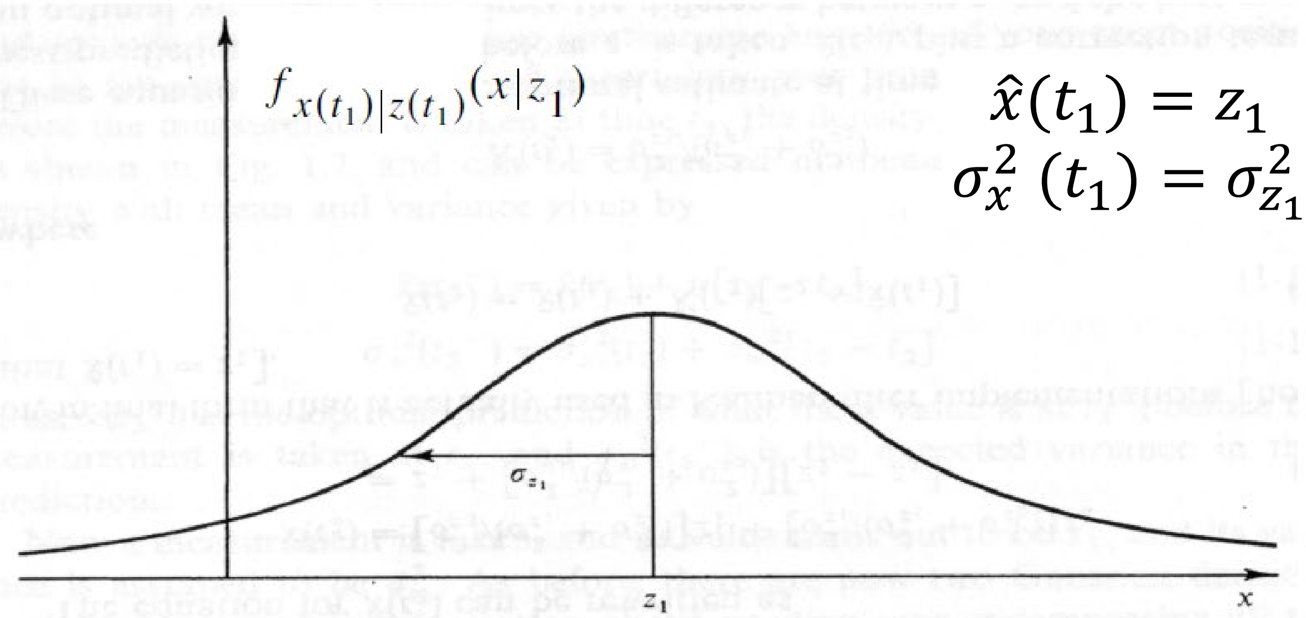
x = 1D position

$\hat{x}$ = 1D position estimate

z = measurement or observation

t_i = time instant i

z_i = measurement taken at t_i

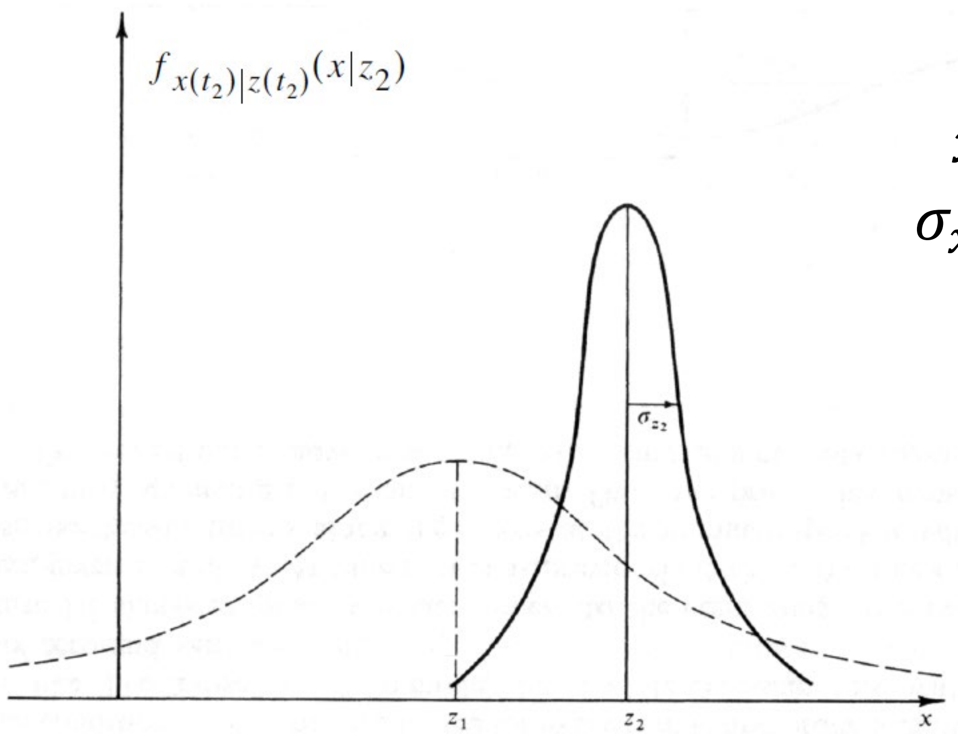


$$\hat{x}(t_1) = z_1$$
$$\sigma_x^2\,(t_1) = \sigma_{z_1}^2$$

[From Maybeck, 1979]

Estimation Based on Static Measurements

- Second measurement taken at $t_2 \approx t_1$
- The smaller σ, the higher the certitude about the measurement



$$\hat{x}(t_2) = z_2$$
$$\sigma_x^2\,(t_2) = \sigma_{z_2}^2$$

[From Maybeck, 1979]

Improving the Estimate Through Fusion

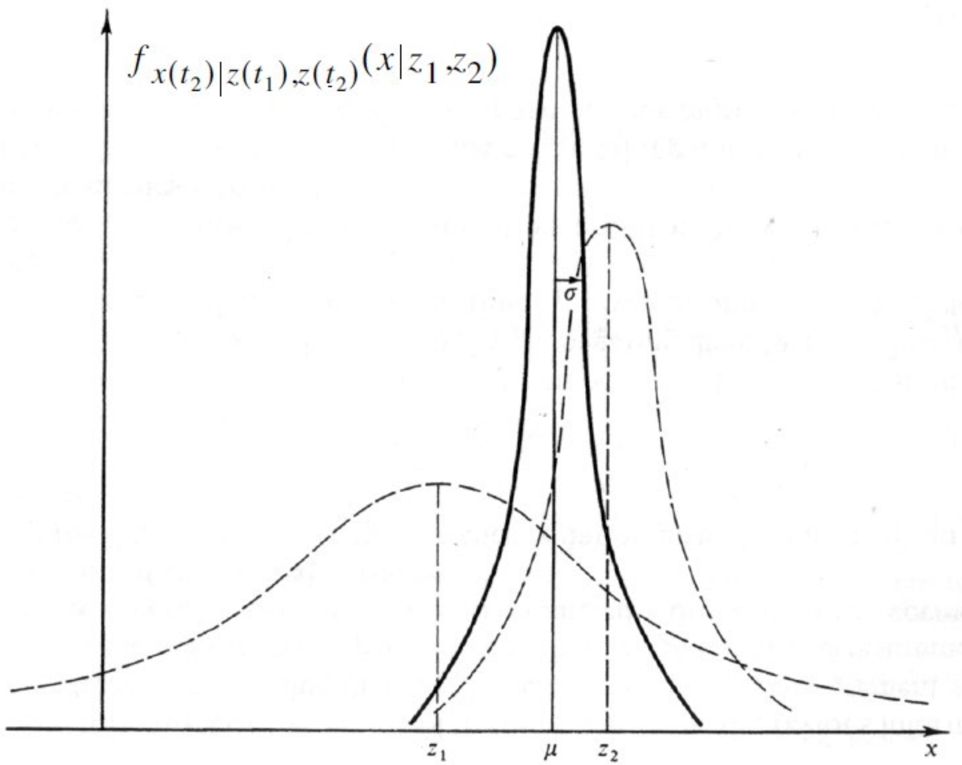
- Intuition: the smaller σ, and thus σ², the higher should be the weight in the fused estimate
- Estimate can be obtained as a weighted average μ of the individual measurement contributions

$$\hat{x}(t_2) = \mu$$
$$\sigma_x(t_2) = \sigma$$

$$\mu = \frac{\frac{1}{\sigma_{z_1}^2} z_1 + \frac{1}{\sigma_{z_2}^2} z_2}{\frac{1}{\sigma_{z_1}^2} + \frac{1}{\sigma_{z_2}^2}}$$

$$\frac{1}{\sigma^2} = \frac{1}{\sigma_{z_1}^2} + \frac{1}{\sigma_{z_2}^2}$$

$$\sigma < \sigma_{z_1} \text{ and } \sigma < \sigma_{z_2}$$



[From Maybeck, 1979]

Mean of the New Position Estimate

$$\hat{x}(t_2) = \frac{\frac{1}{\sigma_{z_1}^2} z_1 + \frac{1}{\sigma_{z_2}^2} z_2}{\frac{1}{\sigma_{z_1}^2} + \frac{1}{\sigma_{z_2}^2}} = \frac{\sigma_{z_2}^2}{\sigma_{z_1}^2 + \sigma_{z_2}^2} z_1 + \frac{\sigma_{z_1}^2}{\sigma_{z_1}^2 + \sigma_{z_2}^2} z_2$$

$$= \frac{\sigma_{z_2}^2}{\sigma_{z_1}^2 + \sigma_{z_2}^2} z_1 \boxed{+ \frac{\sigma_{z_1}^2}{\sigma_{z_1}^2 + \sigma_{z_2}^2} z_1 - \frac{\sigma_{z_1}^2}{\sigma_{z_1}^2 + \sigma_{z_2}^2} z_1} + \frac{\sigma_{z_1}^2}{\sigma_{z_1}^2 + \sigma_{z_2}^2} z_2$$

$$= \frac{\sigma_{z_2}^2 + \sigma_{z_1}^2}{\sigma_{z_1}^2 + \sigma_{z_2}^2} z_1 + \frac{\sigma_{z_1}^2}{\sigma_{z_1}^2 + \sigma_{z_2}^2} (z_2 - z_1)$$

$$= z_1 + \frac{\sigma_{z_1}^2}{\sigma_{z_1}^2 + \sigma_{z_2}^2} (z_2 - z_1)$$

Kalman filter formulation

$$\hat{x}(t_2) = \hat{x}(t_1) + K(t_2)[z_2 - \hat{x}(t_1)]$$

with $K(t_2) = \frac{\sigma_{z_1}^2}{\sigma_{z_1}^2 + \sigma_{z_2}^2}$

$\hat{x}(t_1) = z_1$

Variance of the New Position Estimate

$$\frac{1}{\sigma_x^2(t_2)} = \frac{1}{\sigma_{z_1}^2} + \frac{1}{\sigma_{z_2}^2}$$

$$\sigma_x^2(t_2) = \frac{\sigma_{z_1}^2 \sigma_{z_2}^2}{\sigma_{z_1}^2 + \sigma_{z_2}^2} = \frac{\sigma_{z_2}^2}{\sigma_{z_1}^2 + \sigma_{z_2}^2}\, \sigma_{z_1}^2 = \frac{\sigma_{z_1}^2 + \sigma_{z_2}^2 - \sigma_{z_1}^2}{\sigma_{z_1}^2 + \sigma_{z_2}^2}\, \sigma_{z_1}^2 = \left(1 - \frac{\sigma_{z_1}^2}{\sigma_{z_1}^2 + \sigma_{z_2}^2}\right) \sigma_{z_1}^2$$

$$= \sigma_{z_1}^2 - \frac{\sigma_{z_1}^2}{\sigma_{z_1}^2 + \sigma_{z_2}^2}\, \sigma_{z_1}^2$$

Kalman filter formulation

$$\sigma_x^2(t_2) = \sigma_x^2\ (t_1)\ - K(t_2)\sigma_x^2(t_1)$$

with $K(t_2) = \dfrac{\sigma_{z_1}^2}{\sigma_{z_1}^2 + \sigma_{z_2}^2}$

$$\sigma_x^2(t_1) = \sigma_{z_1}^2$$

Kalman Filter for Sensor Fusion

$$\hat{x}(t_2) = \hat{x}(t_1) + K(t_2)[z_2 - \hat{x}(t_1)]$$

Mean

Prediction

Correction or Update

$$\sigma_x^2(t_2) = \sigma_x^2(t_1) - K(t_2)\,\sigma_x^2(t_1)$$

Variance

$$K(t_2) = \frac{\sigma_{z_1}^2}{\sigma_{z_1}^2 + \sigma_{z_2}^2}$$

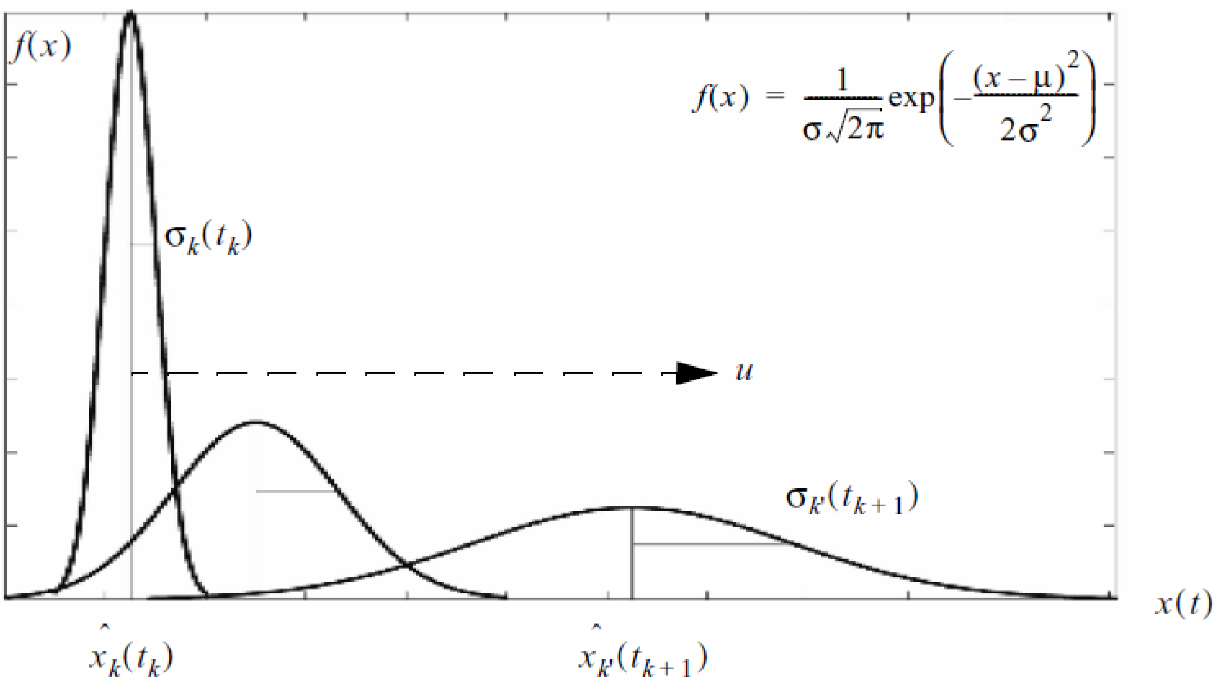
Kalman Gain

Function of the sensing precision

Estimation Considering Motion Dynamics

Consider a simple but noisy motion model: $\dot{x} = u + w$

u = constant speed (controllable input)
w ~ $\mathcal{N}(0, \sigma_w^2)$ = white Gaussian motion noise of mean 0 and variance σ_w^2



[From Siegwart and Nourbakhsh, 2004, adapted from Maybeck 1979]

σ_k^2: position variance at timestep k (known)

$\sigma_{k'}^2$: position variance at timestep k+1

Estimation Based on Motion Model

$$\hat{x}_{k'} = \hat{x}_k + u[t_{k+1} - t_k]$$

- New position mean at timestep t_{k+1}
- Can be estimated with deterministic displacement from motion model

$$\sigma_{k'}^2 = \sigma_k^2 + \sigma_w^2[t_{k+1} - t_k]$$

- New position variance at timestep t_{k+1}
- Motion noise (constant over time) cumulates over time and result in enlarged position variance

Fusing Motion Model Prediction with New Measurement - Mean

$$\hat{x}_{k+1} = \hat{x}_{k'} + K_{k+1}(z_{k+1} - \hat{x}_{k'})$$ with $K_{k+1} = \frac{\sigma_{k'}^2}{\sigma_{k'}^2 + \sigma_z^2}$

Prediction based on motion model

Correction or update based on observation

z_{k+1} : measurement at timestep k+1

K_{k+1} : Kalman gain at timestep k+1

$\hat{x}_{k+1}$: new position estimate at timestep k+ 1 incorporating observation and prediction of motion model

$\hat{x}_{k'}$: position estimate just before timestep k+1 based on prediction motion model

Fusing Motion Model Prediction with New Measurement - Variance

$$\sigma_{k+1}^2 = \sigma_{k'}^2 - K_{k+1}\,\sigma_{k'}^2 \qquad \text{with } K_{k+1} = \frac{\sigma_{k'}^2}{\sigma_{k'}^2 + \sigma_z^2}$$

Prediction based on motion model

Correction or update based on observation

K_{k+1} : Kalman gain at timestep k+1

σ_{k+1}^2 : position variance at timestep k+ 1 incorporating correction from observation and prediction of motion model

$\sigma_{k'}^2$: position variance just before timestep k+1 based on prediction motion model

σ_z^2 : measurement variance (constant over time)

Kalman Filter for Sensor and Motion Model Fusion

$$\hat{x}_{k+1} = \hat{x}_{k'} + K_{k+1}(z_{k+1} - \hat{x}_{k'})$$

Mean

Prediction

Correction or Update

$$\sigma^2_{k+1} = \sigma^2_{k'} - K_{k+1}\,\sigma^2_{k'}$$

Variance

$$K_{k+1} = \frac{\sigma^2_{k'}}{\sigma^2_{k'} + \sigma^2_z}$$

Kalman Gain

Function of the motion model and sensing precision

Kalman Filter - Some Extreme Cases

$$\hat{x}_{k+1} = \hat{x}_{k'} + K_{k+1}(z_{k+1} - \hat{x}_{k'})$$

$$\sigma^2_{k+1} = \sigma^2_{k'} - K_{k+1}\,\sigma^2_{k'}$$

$$K_{k+1} = \frac{\sigma^2_{k'}}{\sigma^2_{k'} + \sigma^2_z}$$

$\sigma_z^2 \rightarrow \infty$: new measurement extremely noisy, does not add information, then $K_{k+1} \rightarrow 0$ new position estimate based exclusively on motion model (both mean and variance)

$\sigma_w^2 \rightarrow \infty$, then $\sigma^2_{k'} \rightarrow \infty$ (see. s. 20): motion model does not add information, then $K_{k+1} \rightarrow 1$, new position estimate based exclusively on new observation

$\sigma_w^2 \rightarrow 0$, then $\sigma^2_{k'} \rightarrow 0$ (see. s. 20, with $\sigma_k^2 = 0$): motion model is deterministic and perfectly reproducing the reality, then $K_{k+1} \rightarrow 0$, new measurement can be disregarded since model is giving a perfect position estimate

The Kalman Filter Algorithm for Multi-Dimension Problems

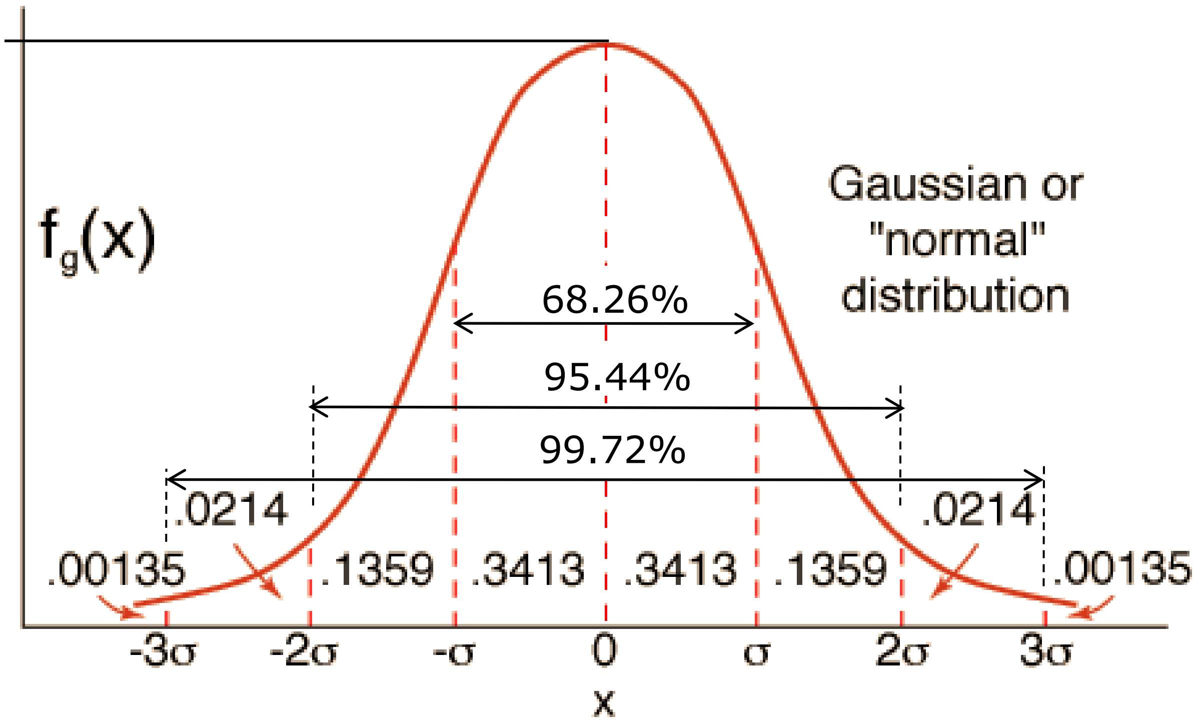
Kalman Filter - Assumptions

- Linear system and measurement model
- White Gaussian system and measurement noise

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) \equiv \mathcal{N}(\mu, \sigma^2)$$

Notes on noise:

- White: uncorrelated in time (used to reproduce a generalized stochastic process in time-discrete systems)
- Gaussian: probability density of amplitude follows bell-shaped curve



x = μ

1D Time-Discrete Motion Model

- Noisy, time-continuous motion model (see s. 19):

$$\dot{x} = u + w$$

x = system state
u = constant speed (controllable input)
$w \sim \mathcal{N}(0, \sigma_w^2)$ = Gaussian motion noise of mean 0 and variance σ_w^2

- Notation change for time-discrete motion model:

$$x_t - x_{t-1} = u_t \Delta t + \varepsilon_t$$

t = time index, implying a certain time-discretization interval Δt
u_t = variable speed assumed constant during Δt (controllable input)
$\varepsilon_t = w\Delta t \sim \mathcal{N}(0, \sigma_w^2 \Delta t)$ Gaussian position noise whose distribution is assumed constant during Δt

1D Time-Discrete Measurement Model

- Noisy, time-continuous measurement model (see s. 13-14):

$$z = x + v$$

x = system state
z = observation
$v \sim \mathcal{N}(0, \sigma_z^2)$ = Gaussian measurement noise of 0 mean and variance σ_z^2

- Notation change for time-discrete measurement model:

$$z_t = x_t + \delta_t$$

t = time index, implying a certain time-discretization interval Δt
$\delta_t = v \sim \mathcal{N}(0, \sigma_z^2)$: Gaussian measurement noise whose distribution is assumed constant during Δt

EPFL

Multi-Dimensional Kalman Filter

Generalized time-discrete equations (difference equations) expressing state and measurement vectors for a stochastic system (process) (corresponding to s. 27 for the specific 1D motion model example):

$$\boldsymbol{x}_t = A_t \boldsymbol{x}_{t-1} + B_t \boldsymbol{u}_t + \boldsymbol{\varepsilon}_t$$

with a measurement

$$\boldsymbol{z}_t = C_t \boldsymbol{x}_t + \boldsymbol{\delta}_t$$

State **vector** x: dim *n*
Input **vector** u: dim *m*
System noise **vector** ε : dim *n*
Measurement **vector** z: dim *k*
Measur. noise **vector** δ : dim *k*

[Adapted from Thrun et al., 2005]

Components of a Kalman Filter

System equation: $\boldsymbol{x_t} = A_t \boldsymbol{x_{t-1}} + B_t \boldsymbol{u_t} + \boldsymbol{\varepsilon_t}$

Measurement equation: $\boldsymbol{z_t} = C_t \boldsymbol{x_t} + \boldsymbol{\delta_t}$

A_t Matrix (n*n) that describes how the state evolves from t-1 to t without controls or noise.

B_t Matrix (n*m) that describes how the control u_t changes the state from t-1 to t.

C_t Matrix (k*n) that describes how to map the state x_t to an observation z_t.

ε_t

δ_t

Gaussian random vectors representing the system and measurement noise that are assumed to be independent and normally distributed with covariance R_t and Q_t, respectively.

[Adapted from Thrun et al., 2005]

Kalman Filter Algorithm

Algorithm **Kalman_filter**($\boldsymbol{x}_{t-1}$, Σ_{t-1}, $\boldsymbol{u}_t$, $\boldsymbol{z}_t$)

Prediction:

1. $\widehat{\boldsymbol{x}}_t = A_t \boldsymbol{x}_{t-1} + B_t \boldsymbol{u}_t$
2. $\widehat{\Sigma}_t = A_t \Sigma_{t-1} A_t^T + R_t$

State prediction from the input and system equation

System covariance

Correction or update:

3. $K_t = \widehat{\Sigma}_t C_t^T (C_t \widehat{\Sigma}_t C_t^T + Q_t)^{-1}$
4. $\boldsymbol{x}_t = \widehat{\boldsymbol{x}}_t + K_t (\boldsymbol{z}_t - C_t \widehat{\boldsymbol{x}}_t)$
5. $\Sigma_t = (I - K_t C_t) \widehat{\Sigma}_t$

Measurement covariance

Update predicted state with measurements

Predicted measurements

Measurements

Return $\boldsymbol{x}_t$, Σ_t

[Adapted from Thrun et al., 2005]

2D Linear Motion Model

- Consider a perfectly holonomic vehicle:

$$x_t = x_{t-1} + u_{x,t}\Delta t + \varepsilon_{x,t}$$
$$y_t = y_{t-1} + u_{y,t}\Delta t + \varepsilon_{y,t}$$

Notes:
- See s. 27 for 1D
- Holonomic: omnidirectional (no motion constraints)

- With symmetric white Gaussian position noise of mean 0 and variance $\sigma_w^2 \Delta t$ in both directions: $\varepsilon_{x,t} = \varepsilon_{y,t} \sim \mathcal{N}(0, \sigma_w^2 \Delta t)$

- Then the KF's variables and parameters are as follows:

$$\boldsymbol{x_t} = \begin{bmatrix} x_t \\ y_t \end{bmatrix} \qquad \boldsymbol{u_t} = \begin{bmatrix} u_{x,t} \\ u_{y,t} \end{bmatrix} \qquad R_t = \begin{bmatrix} \sigma_w^2 \Delta t & 0 \\ 0 & \sigma_w^2 \Delta t \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} \Delta t & 0 \\ 0 & \Delta t \end{bmatrix} \qquad \hat{\Sigma}_t = A_t \Sigma_{t-1} A_t^T + R_t$$

2D Linear Measurement Model

- Consider a measurement of x, y position

$$z_{x,t} = x_t + \delta_{x,t}$$
$$z_{y,t} = y_t + \delta_{y,t}$$

Note: measurement for instance obtained through a global positioning system such as a GNSS

- With symmetric Gaussian measurement noise of mean 0 and variance σ_z^2 in both directions: $\delta_{x,t} = \delta_{y,t} \sim \mathcal{N}(0, \sigma_z^2)$

- Then the KF's variables and parameters are as follows:

$$\boldsymbol{x_t} = \begin{bmatrix} x_t \\ y_t \end{bmatrix} \qquad \boldsymbol{z_t} = \begin{bmatrix} z_{x,t} \\ z_{y,t} \end{bmatrix} \qquad Q_t = \begin{bmatrix} {\sigma_z}^2 & 0 \\ 0 & {\sigma_z}^2 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad K_t = \hat{\Sigma}_t C_t^T (C_t \hat{\Sigma}_t C_t^T + Q_t)^{-1}$$

Conclusion

Take Home Messages

- Feature-based localization is a way to compensate odometry limitations such as cumulative error by leveraging exteroceptive sensors in addition to the proprioceptive ones used for odometry
- A Kalman filter is a computationally efficient, optimal recursive data processing algorithm that allows fusion of multiple estimates coming from either process (system, motion) or sensing (measurement) models
- A Kalman filter assumes linear motion and measurement models characterized by white Gaussian noise
- A Kalman filter can be used for solving multi-dimensional problems, including 2D and 3D localization in mobile robotics

Additional Literature – Week 10

Books

- Weston J. and Titterton D, “Strapdown Inertial Navigation”, IET, 2005
- Siegwart R., Nourbakhsh I., and Scaramuzza D., “Introduction to Autonomous Mobile Robots, second Edition”, MIT Press, 2011.
- Borenstein J., Everett H. R., and Feng L. “Navigating Mobile Robots: Systems and Techniques”, A. K. Peters, Ltd., 1996.
- Thrun S., Burgard W., and Fox D., Probabilistic Robotics, MIT Press, 2005.
- Choset H., Lynch K. M., Hutchinson S., Kantor G., Burgard W., Kavraki L., and Thrun S., “Principles of Robot Motion”. MIT Press, 2005.
- Simon D., “Optimal State Estimation: Kalman, H∞, and Nonlinear Approaches”, John Wiley & Son, 2006.