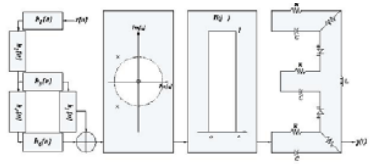
Signals, Instruments, and Systems – W4

Introduction to Signal Processing – System Properties, Laplace Transform, Responses, Functions and Filters in Continuous Time

Outline

- System properties
- Laplace transform
- In continuous time:
  - Frequency responses and transfer functions
  - Motivation for filters and relation with systems
  - Filter terminology and basic examples
  - Examples of simple analog filters

Acknowledgment for Selected Slides



Signals and Systems

Fall 2003

Lecture #1

Prof. Alan S. Willsky

4 September 2003

1) Administrative details
2) Signals
3) Systems
4) For examples ...

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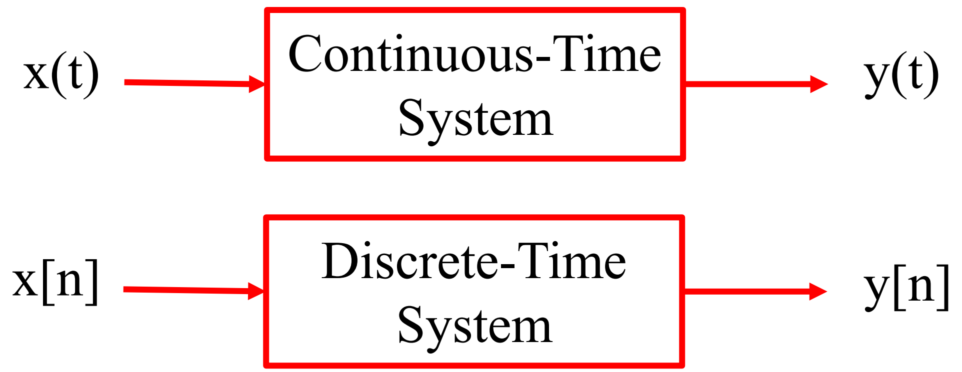
1

System Properties

System Properties



- Knowing what system properties are fulfilled helps the analysis of the system and has practical implications
- Three key properties: causality, time-invariance, and linearity

Causality

CAUSALITY

- A system is causal if the output does not anticipate future values of the input, i.e., if the output at any time depends only on values of the input up to that time.

- All real-time physical systems are causal, because time only moves forward. Effect occurs after cause. (Imagine if you own a noncausal system whose output depends on tomorrow's stock price.)

- Causality does not apply to spatially varying signals. (We can move both left and right, up and down.)

- Causality does not apply to systems processing recorded signals, e.g. taped sports games vs. live broadcast.

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Causality

CAUSALITY (continued)

- Mathematically (in CT): A system $x(t) \to y(t)$ is causal if

when	$x_1(t) \to y_1(t)$	$x_2(t) \to y_2(t)$
and	$x_1(t) = x_2(t)$	for all $t \leq t_o$
Then	$y_1(t) = y_2(t)$	for all $t \leq t_o$

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Time-Invariance

TIME-INVARIANCE (TI)

Informally, a system is time-invariant (TI) if its behavior does not depend on what time it is.

- Mathematically (in DT): A system $x[n] \rightarrow y[n]$ is TI if for <u>any</u> input $x[n]$ and <u>any</u> time shift n_0,

  If $\quad x[n] \rightarrow y[n]$

  then $\quad x[n - n_0] \rightarrow y[n - n_0]$.

- Similarly for a CT time-invariant system,

  If $\quad x(t) \rightarrow y(t)$

  then $\quad x(t - t_o) \rightarrow y(t - t_o)$.

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Linearity

A (CT) system is linear if it has the superposition property:

If $\quad x_1(t) \rightarrow y_1(t) \quad$ and $\quad x_2(t) \rightarrow y_2(t)$

then $\quad ax_1(t) + bx_2(t) \quad \rightarrow \quad ay_1(t) + by_2(t)$

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Additional Definitions

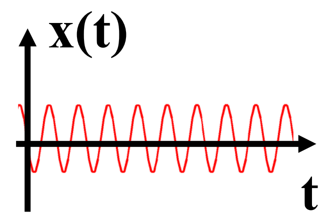
- LTI: Linear Time-Invariant systems
- SISO: Single-Input Single-Output system
- MIMO: Multi-Input Multi-Output system

In this course, we will consider mainly SISO systems (e.g., a SISO filter) in an operational regime respecting the LTI properties

Impulse Response of LTI Systems and Convolution

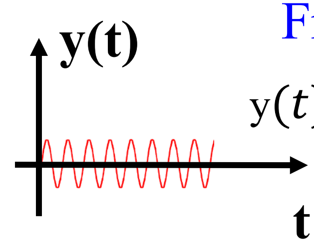
Perturbations and Responses of a LTI System

$x(t)$ → **Continuous-Time System** → $y(t)$

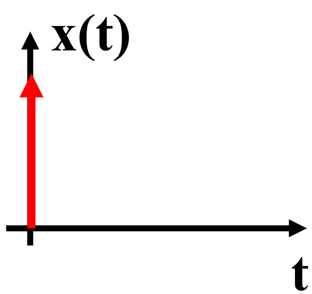


$x(t) = A_i \sin(\omega t + \varphi_i)$

Frequency response

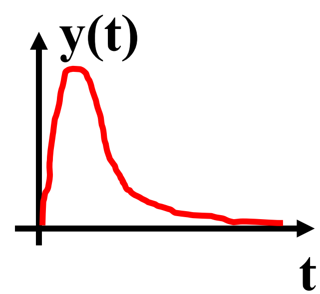


$y(t) = A_o \sin(\omega t + \varphi_o)$

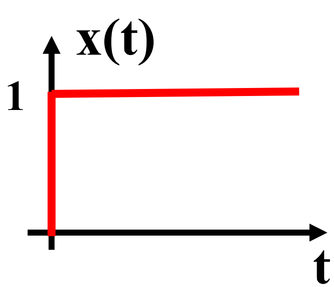


$x(t) = \delta(t)$

Impulse response

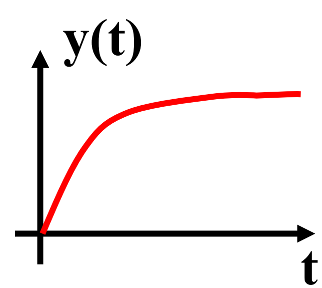


$y(t) = h(t)$



$x(t) = u(t)$

Step response

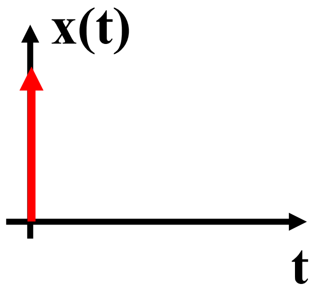


$y(t) = y_u(t)$

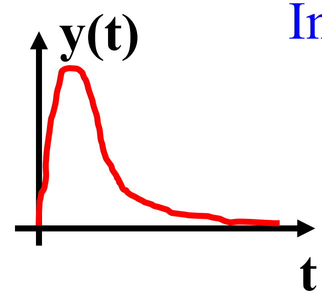
Perturbations and Responses of a LTI System

Some important notes on previous slide:

- Similar definitions and examples in discrete time
- Physically speaking, it is easier to generate a step function than an impulse
- The impulse response is the time derivative of the step response or, equivalently, the step response is the running integral over time of the impulse response

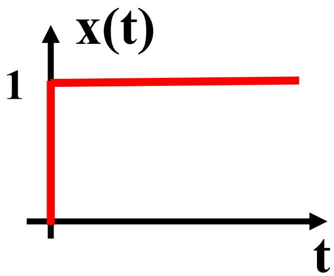


$$x(t) = \delta(t)$$

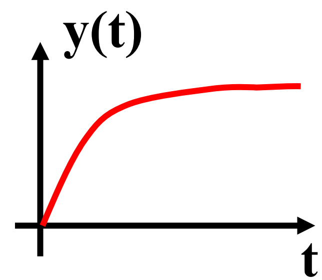


Impulse response

$$y(t) = h(t)$$



$$x(t) = u(t)$$



Step response

$$y(t) = y_u(t)$$

Convolution with Dirac Impulses in Continuous and Discrete time

Associated to this book:
J. H. McClellan, R. W. Schafer, M. A. Yoder
"DSP First: A Multimedia Approach", Prentice Hall, 1999.

Available for download here and on Moodle (Week2):

https://dspfirst.gatech.edu/matlab/

Convolution: From Mathematical Definition to LTI System Responses

From W2 definitions, s. 39 (continuous convolution) and s. 48 (discrete convolution)

$f(t) = \delta(\mathrm{t}) \quad \Rightarrow (f * h)(t) = h(t)$

$f[n] = \delta[\mathrm{n}] \quad \Rightarrow (f * h)[\mathrm{n}] = h[n]$

Note: See also W2 s. 22, 43, 49 for graphical representations

From W3, s. 6: any arbitrary time-discrete signal (function) can be represented by a train (sequence) of weighted Dirac impulses

By analogy, we can think an arbitrary time-continuous signal (function) as being represented by a dense sequence of idealized impulses with vanishing small duration (dense sequence of weighted Dirac impulses)

Convolution Sum and Integral

For time-discrete LTI systems, by exploiting the **time-invariance** and **linearity** properties and assuming that **we can represent any arbitrary input function by a sequence of weighted impulses**, we can demonstrate that the output function of the LTI system is represented by the convolution sum (convolution of the input function and the impulse response of the LTI system)

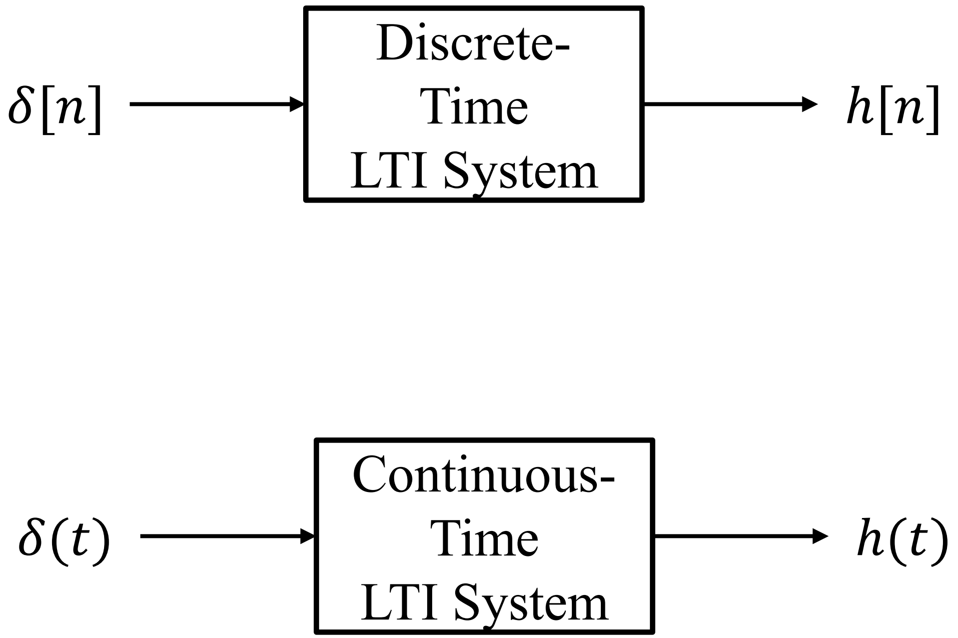
$$f[n] \longrightarrow \boxed{\quad h[n] \quad} \longrightarrow (f * h)[n] = \sum_{m=-\infty}^{\infty} f[m]h[n-m]$$

For time-continuous LTI systems, by analogy, we can demonstrate that the output function of the LTI system is represented by the convolution integral (convolution of the input function and the impulse response of the LTI system)

$$f(t) \longrightarrow \boxed{\quad h(t) \quad} \longrightarrow (f * h)(t) = \int_{-\infty}^{\infty} f(\tau)h(t-\tau)d\tau$$

$\boxed{\text{See also derivation of the Convolution Theorem (downloadable from Moodle)}}$

Impulse Response of a LTI System



By perturbing a LTI system with an impulse, we obtain the impulse response, which fully characterizes the LTI system

Impulse vs. Step Response in Time Domain

Based on the convolution properties, we can formally revisit the statement that the impulse response is the time derivative of the step response or, equivalently, the step response is the running integral over time of the impulse response

$\delta(t) = \text{impulse}$

$u(t) = \text{step function (Heaviside step function)}$

$$\dot{u}(t) = \frac{du(t)}{dt} = \delta(t)$$

Step response

$$u(t) \longrightarrow \boxed{h(t)} \longrightarrow (u * h)(t)$$

Can be proved with convolution properties

$$\delta(t) \longrightarrow \boxed{h(t)} \longrightarrow (\delta * h)(t) = (\frac{du(t)}{dt} * h)(t) = \frac{d}{dt}(u * h)(t)$$

Impulse response

Laplace Transforms

Motivation

Motivation for the Laplace Transform

- CT Fourier transform enables us to do a lot of things, e.g.
  - Analyze frequency response of LTI systems
  - Sampling
  - Modulation

  ⋮
- Why do we need yet another transform?
- One view of Laplace Transform is as an *extension* of the Fourier transform to allow analysis of broader class of signals and systems
- In particular, Fourier transform *cannot* handle large (and important) classes of signals and *unstable* systems, i.e. when

$$\int_{-\infty}^{\infty} |x(t)|dt = \infty$$

Note: compare with W2, s.19

From Prof. A. S. Willsky, Signals and Systems course

Laplace Transform

$$F(s) = \mathcal{L}\{f(t)\} = \int_{-\infty}^{\infty} e^{-st} f(t) dt$$

$$s = \sigma + i\omega$$

- Transform signals from time-domain to a complex frequency-domain representation (s-plane)
- The standard Laplace transform is involving an integral between 0 and ∞; the bounds - ∞ to ∞ are used for the bilateral Laplace transform which is what we will use in this course (calling by simplicity Laplace transform)

Fourier - Laplace

$$F(\omega) = \mathcal{F}\{f(t)\} = \mathcal{L}\{f(t)\}\,|_{s=i\omega}$$

$$= F(s)\,|_{s=i\omega} = \int_{-\infty}^{\infty} e^{-i\omega t} f(t)\,dt$$

Note: non-unitary, angular frequency, see W2, s. 21

The Fourier Transform is therefore a special case of the Laplace Transform

Fourier: frequency response (in stationary conditions, especially in signal processing)
Laplace: impulse response (also in transient conditions, especially in control)

EPFL

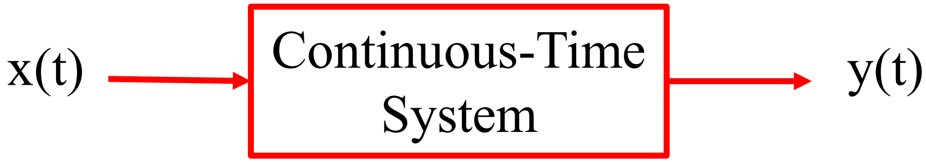
Disal

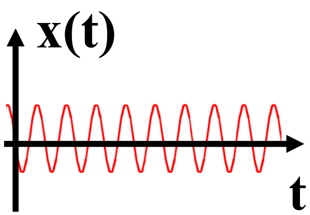
Some Properties of Laplace Transform

Property	Signal	Laplace Transform
	$x(t)$	$X(s)$
	$x_1(t)$	$X_1(s)$
	$x_2(t)$	$X_2(s)$
Linearity	$ax_1(t) + bx_2(t)$	$aX_1(s) + bX_2(s)$
Time shifting	$x(t - t_0)$	$e^{-st_0}X(s)$
Shifting in the s-Domain	$e^{s_0 t}x(t)$	$X(s - s_0)$
Time scaling	$x(at)$	$\frac{1}{\|a\|}X\left(\frac{s}{a}\right)$
Conjugation	$x^*(t)$	$X^*(s^*)$
Convolution	$x_1(t) * x_2(t)$	$X_1(s)X_2(s)$
Differentiation in the Time Domain	$\frac{d}{dt}x(t)$	$sX(s)$
Differentiation in the s-Domain	$-tx(t)$	$\frac{d}{ds}X(s)$
Integration in the Time Domain	$\int_{-\infty}^{t} x(\tau)d(\tau)$	$\frac{1}{s}X(s)$

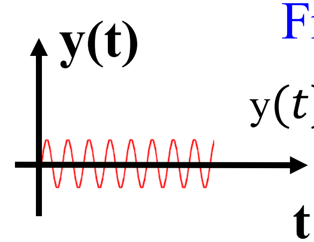
Frequency Responses and Transfer Functions in Continuous Time

Perturbations and Responses of a LTI System



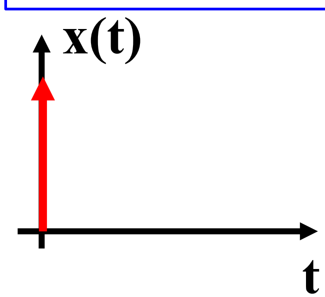


$x(t) = A_i \sin(\omega t + \varphi_i)$

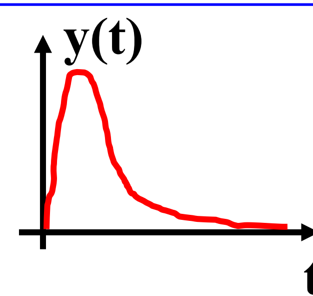


Frequency response

$y(t) = A_o \sin(\omega t + \varphi_o)$

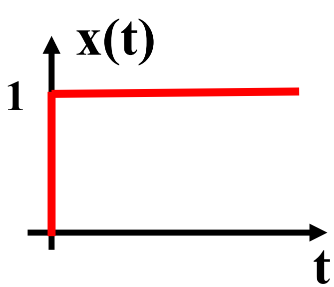


$x(t) = \delta(t)$

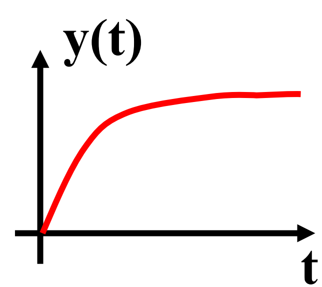


Impulse response

$y(t) = h(t)$



$x(t) = u(t)$



Step response

$y(t) = y_u(t)$

Frequency Response

h(t) : impulse response
H(ω) : frequency response (stationary regime) of the filter/system, i.e Fourier transform of h(t)

Time domain

$$f(t) \longrightarrow \boxed{h(t)} \longrightarrow (f * h)(t)$$

Frequency domain

$$F(\omega) \longrightarrow \boxed{H(\omega)} \longrightarrow F(\omega)H(\omega)$$

Reminder: a convolution in the time domain is a multiplication in the frequency domain

Frequency Response

$$
\begin{tikzcd}
e^{i\omega_0 t} \arrow[r] \arrow[d, "\mathcal{F}"] & \boxed{h(t)} \arrow[r] & H(\omega_0)e^{i\omega_0 t} \\
2\pi\delta(\omega - \omega_0) \arrow[r] & \boxed{H(\omega)} \arrow[r] & H(\omega)2\pi\delta(\omega - \omega_0) = H(\omega_0)2\pi\delta(\omega - \omega_0) \arrow[u, "\mathcal{F}^{-1}"]
\end{tikzcd}
$$

Note: see also s. 13, W3

A single frequency comes out of a LTI filter multiplied with the value of the filter at that frequency.

Frequency Response

Continuous time

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{ik\omega_0 t} \longrightarrow \boxed{h(t)} \longrightarrow y(t) = \sum_{k=-\infty}^{\infty} H(k\omega_0) a_k e^{ik\omega_0 t}$$

$$a_k \rightarrow H(k\omega_0) a_k$$

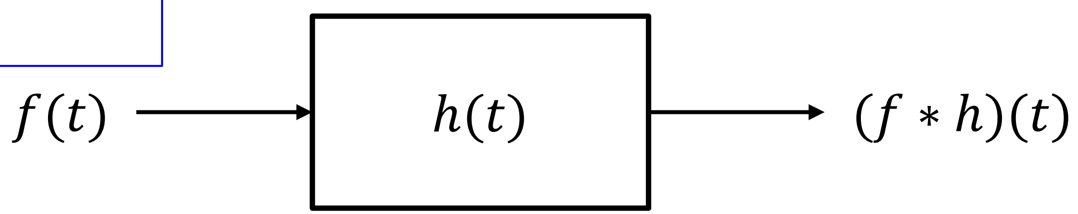
$$H(k\omega_0) = \underbrace{|H(k\omega_0)|}_{\text{Magnitude or Amplitude or Gain}} e^{\overbrace{i\angle H(k\omega_0)}^{\text{Phase}}}$$

By linearity a sum of frequencies go out of the LTI filter only with different amplitude and phase.

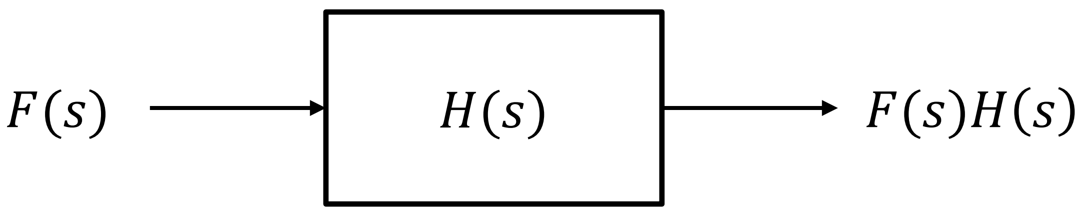
Time-Continuous Transfer Function

h(t) : impulse response (CT)
H(s) : transfer function
(stationary and transient frequency response) of the filter/system, i.e Laplace transform of h(t)

Time domain



Complex frequency domain (s-plane)



Reminder: a convolution in the time domain is a multiplication in the frequency domain

Impulse vs. Step Response in Frequency Domain

Using the Laplace transform, we can formally revisit, even in a simpler way than in time domain, the statement that the impulse response is the time derivative of the step response or, equivalently, the step response is the running integral over time of the impulse response also in the complex s-plane

$\Delta(s)$ = Laplace transform of the impulse $(= 1)$

$\mathrm{U}(s)$ = Laplace transform of the step function $(= \frac{1}{s})$

$$\Delta(s) = sU(s)$$

Step response

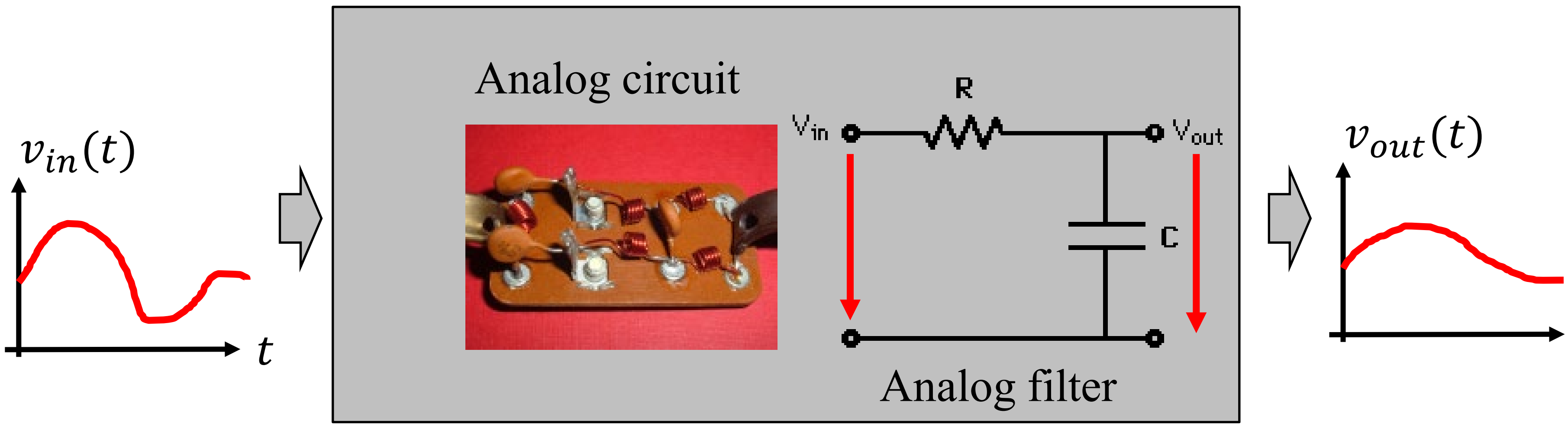
$$U(s) \longrightarrow \boxed{H(s)} \longrightarrow U(s)H(s)$$

$$\Delta(s) \longrightarrow \boxed{H(s)} \longrightarrow \Delta(s)H(s) = [sU(s)]H(s) = s[U(s)H(s)]$$

Impulse response

Filters

Filters as System Examples

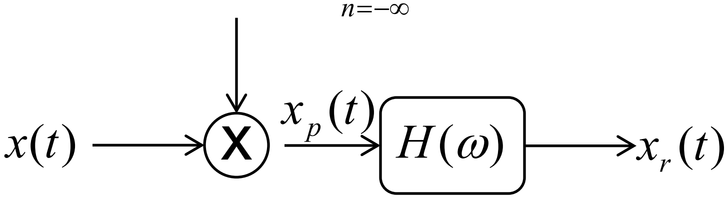


$$v_{in}(t) \longrightarrow \boxed{h(t)} \longrightarrow v_{out}(t)$$

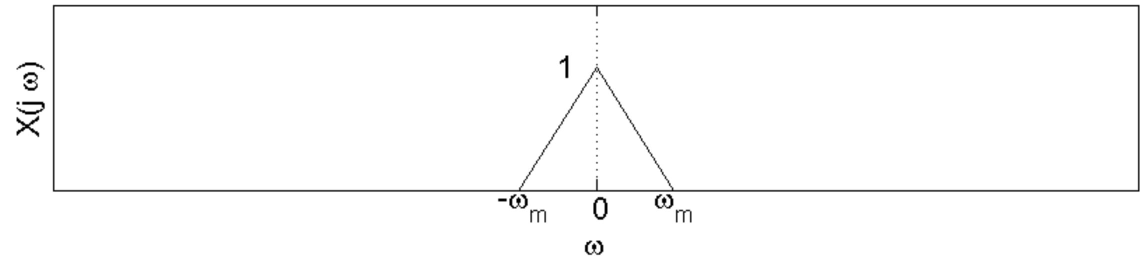
Filters for Signal Reconstruction

From W3, s. 27

$$p(t) = \sum_{n=-\infty}^{+\infty} \delta(t - nT)$$



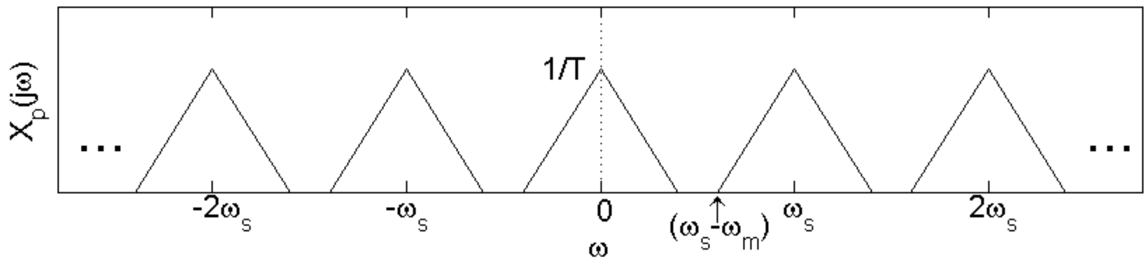
spectrum of original signal



spectrum of sampled signal

sampling angular frequency

$\omega_s > 2\,\omega_m$

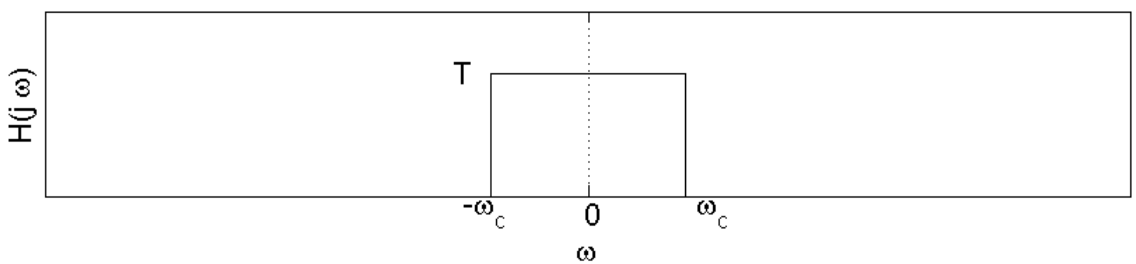


filtering

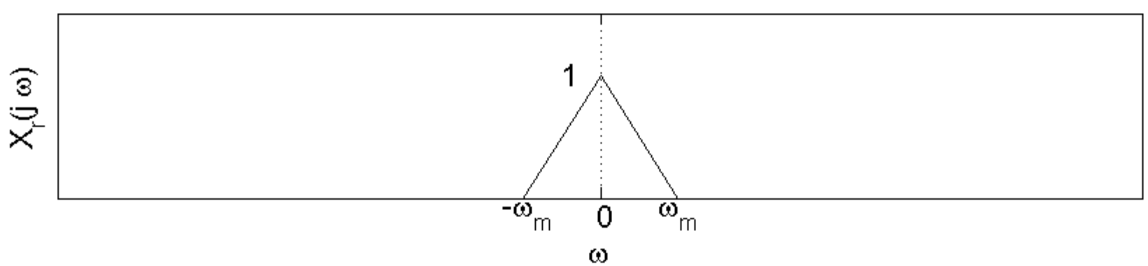
filter cut-off angular frequency

$\omega_m < \omega_c < (\omega_s - \omega_m)$

$$X_r(\omega) = X_p(\omega) H(\omega)$$



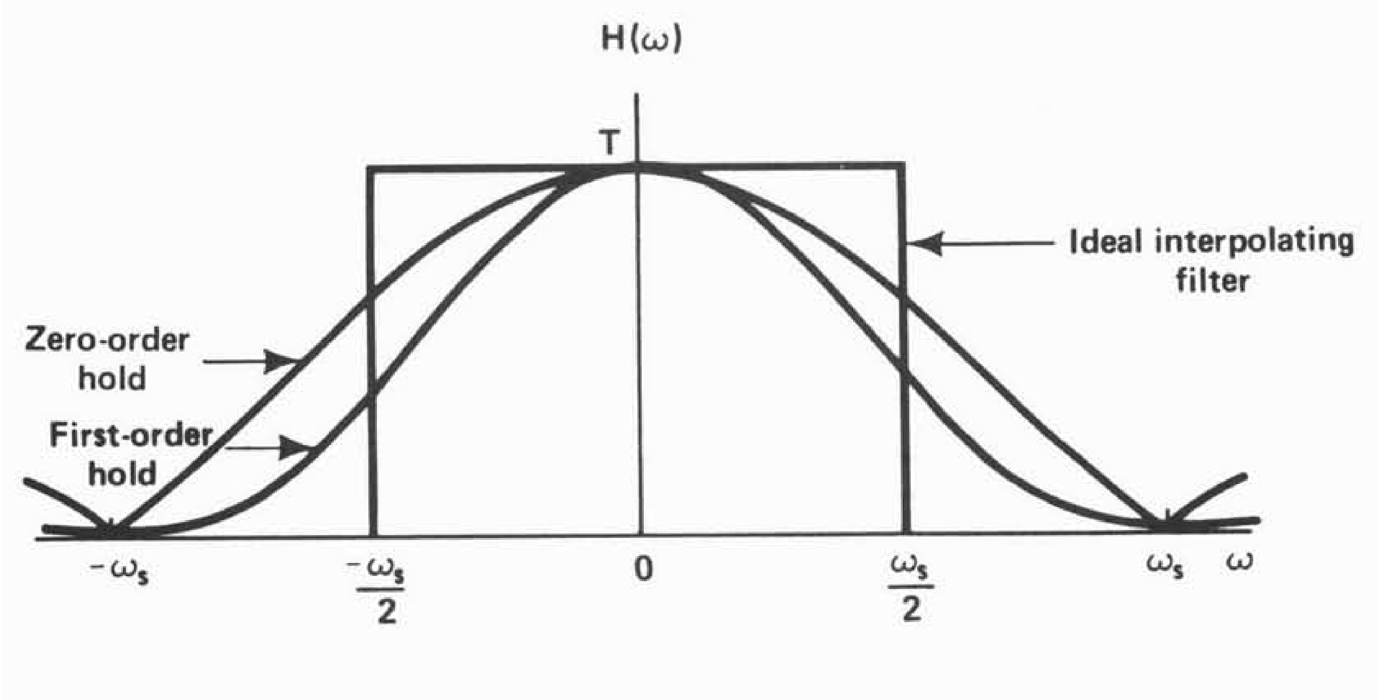
spectrum of reconstructed signal



Reconstruction Summary – Frequency Domain

From W3, s. 34

The spectrum of the reconstructed signal $X_r(\omega)$ is obtained through a multiplication between the spectrum of the sampled signal $X_p(\omega)$ with angular sampling frequency ω_s and one of the following three low-pass filters.



From Prof. A. Oppenheim, Signals and Systems course

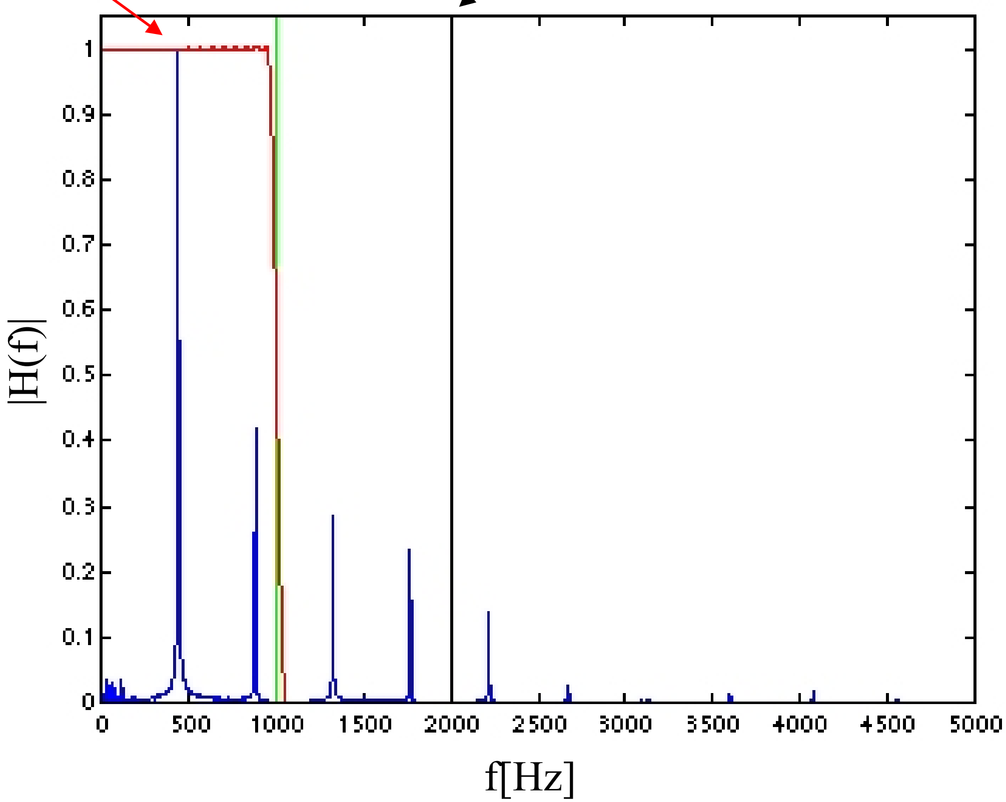
Anti-Aliasing Filters

Actual high-order filter (see W5)

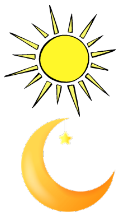
Anti-alias filter: desired cut-off frequency 1 kHz

Reduced sampling frequency: 2 kHz

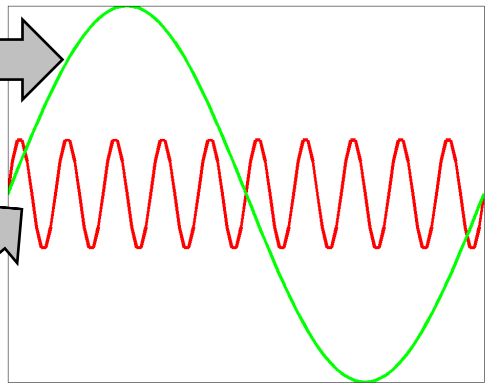
From W3, s. 43

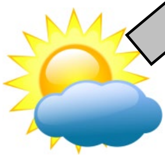


Filtering Noisy Signals

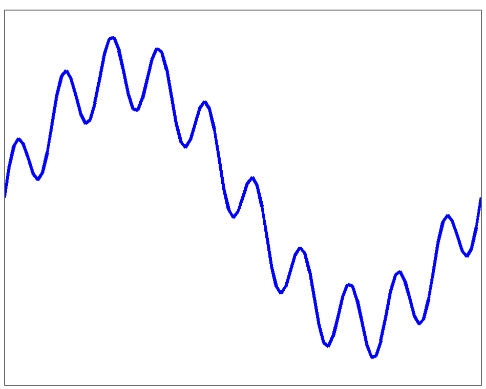


day/night cycle

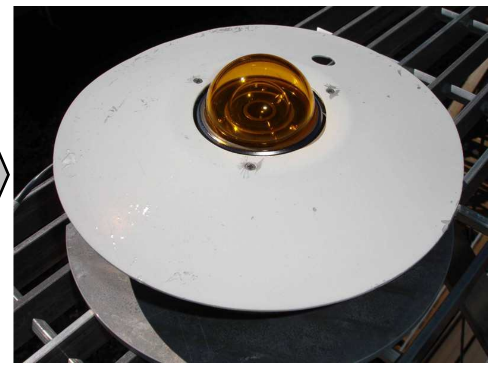


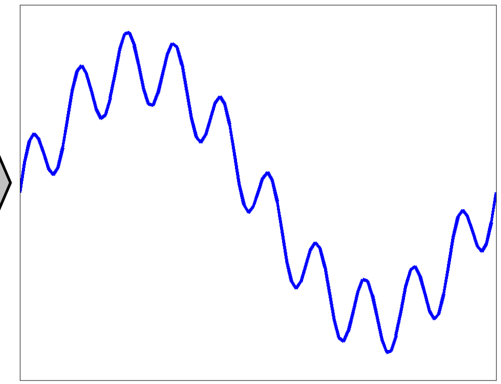


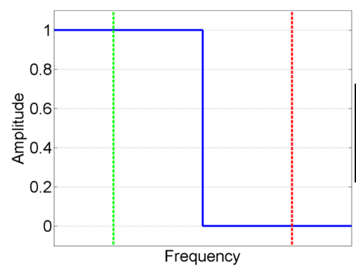
changing cloud cover



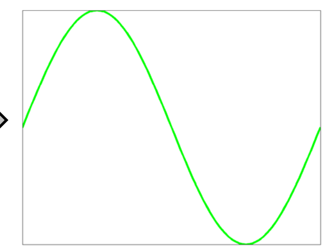
Solar radiation

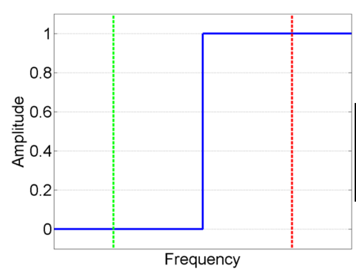




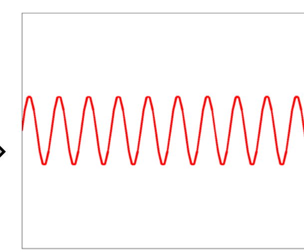


low pass filter





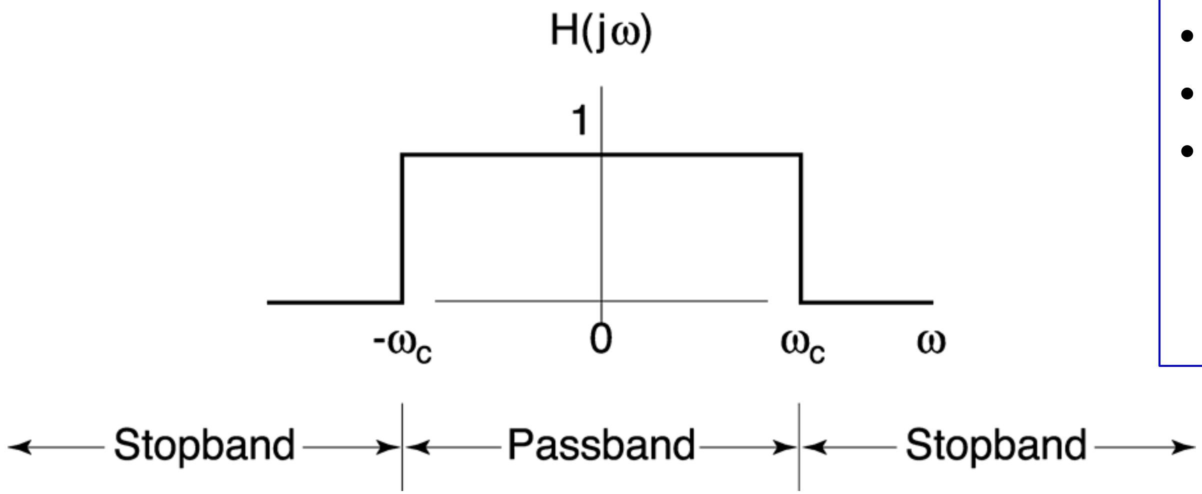
high pass filter



Terminology and Basic Filters

Low-Pass Filter

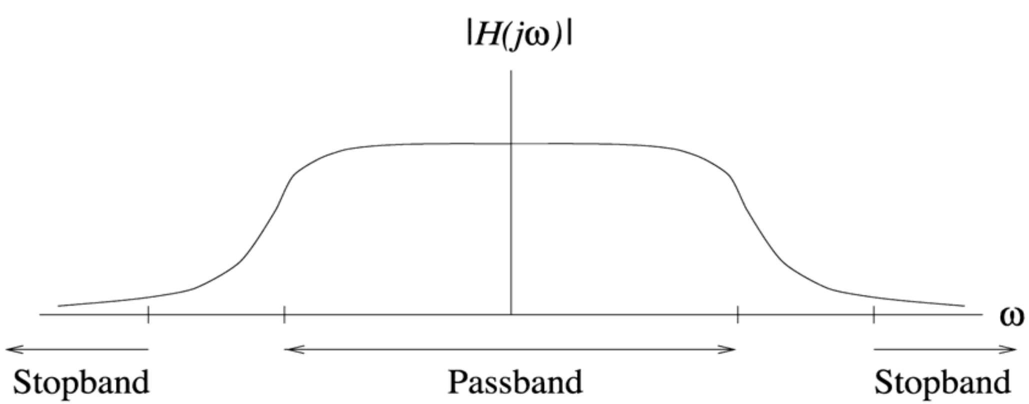
Idealized response in the (angular) frequency domain:



Notes:

- $H(j\omega) = H(i\omega) = H(\omega)$
- ω_c: cut-off frequency
- $|H| = 1$ and $\angle H = 0$ for ideal filters in the passband, no need for the phase plot.

Realistic response in the (angular) frequency domain:

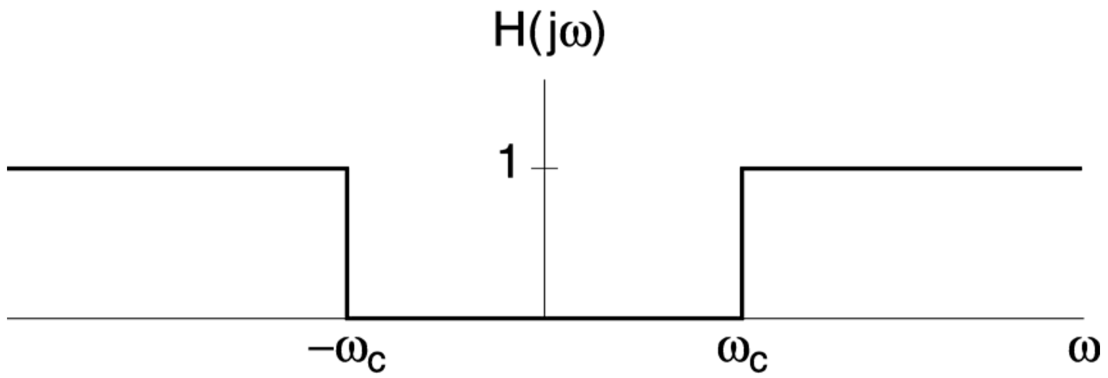


Notes:

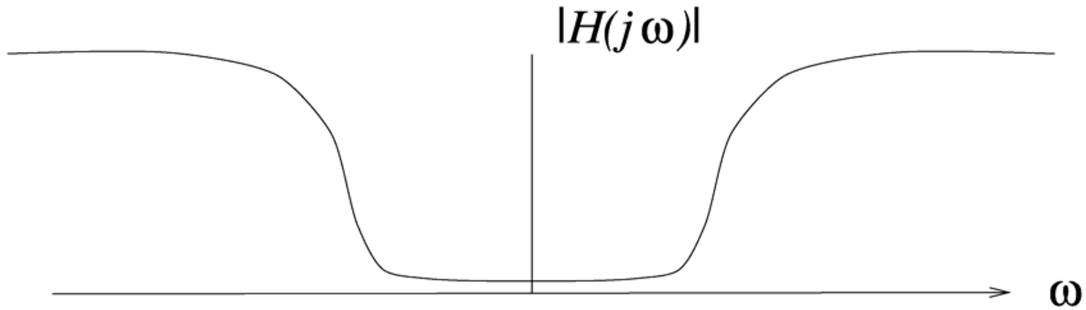
$|H(j\omega)|$: magnitude

High-Pass Filter

Idealized response in the (angular) frequency domain:

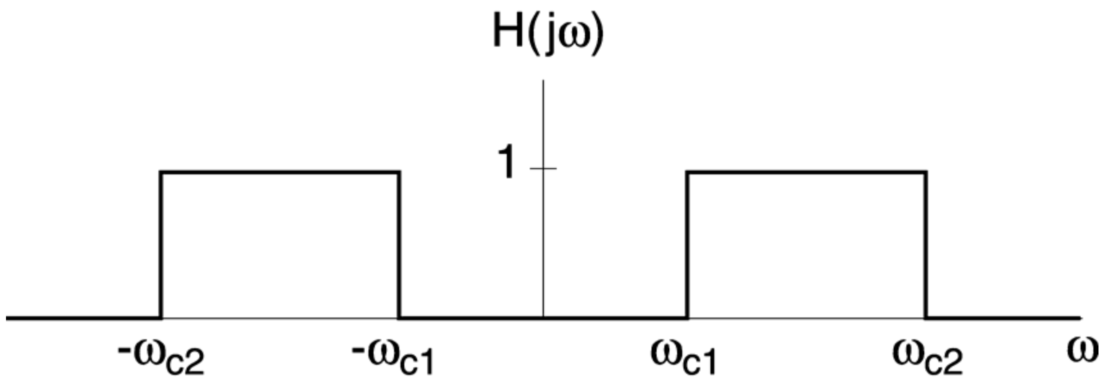


Realistic response in the (angular) frequency domain:



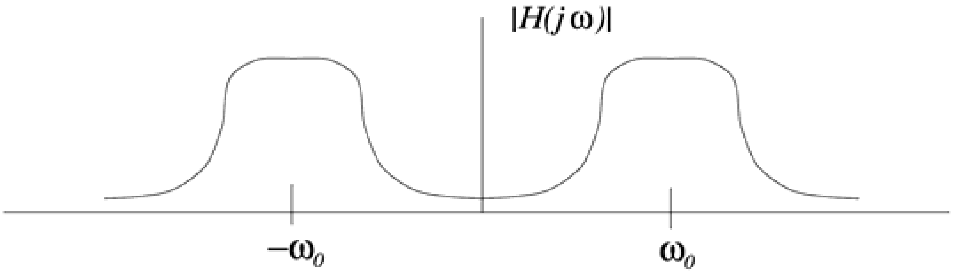
Band-Pass Filter

Idealized response in the (angular) frequency domain:



ω_{c1}: lower cut-off frequency
ω_{c2}: upper cut-off frequency

Realistic response in the (angular) frequency domain:



Notes:
a Band-Stop Filter reverses passing and stopping bands w.r.t. a band-pass filter

Examples of Simple Analog Filters

Decibel

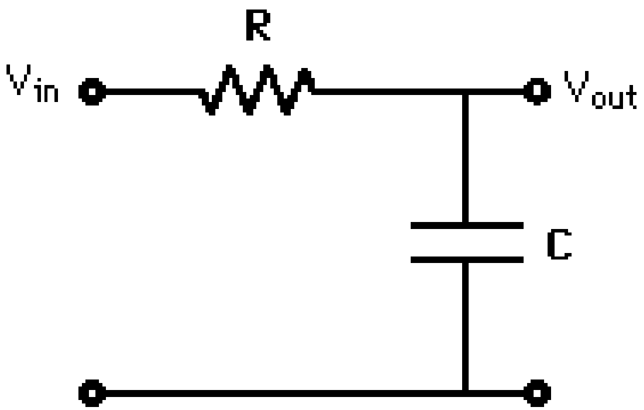
$$G_{dB} = 20\log_{10}\left(\frac{V_2}{V_1}\right)$$

$$V_2 > V_1 \rightarrow G_{dB} > 0\ (gain)$$

$$V_2 < V_1 \rightarrow G_{dB} < 0\ (damping)$$

Source of sound	Sound pressure	Sound pressure level
	pascal	dB re 20 μPa
Jet engine at 30 m	630 Pa	150 dB
Rifle being fired at 1 m	200 Pa	140 dB
Threshold of pain	100 Pa	130 dB
Hearing damage (due to short-term exposure)	20 Pa	approx. 120 dB
Jet at 100 m	6 – 200 Pa	110 – 140 dB
Jack hammer at 1 m	2 Pa	approx. 100 dB
Hearing damage (due to long-term exposure)	6×10^{-1} Pa	approx. 85 dB
Major road at 10 m	2×10^{-1} – 6×10^{-1} Pa	80 – 90 dB
Passenger car at 10 m	2×10^{-2} – 2×10^{-1} Pa	60 – 80 dB
TV (set at home level) at 1 m	2×10^{-2} Pa	approx. 60 dB
Normal talking at 1 m	2×10^{-3} – 2×10^{-2} Pa	40 – 60 dB
Very calm room	2×10^{-4} – 6×10^{-4} Pa	20 – 30 dB
Leaves rustling, calm breathing	6×10^{-5} Pa	10 dB
Auditory threshold at 1 kHz	2×10^{-5} Pa	0 dB

Analog Low-Pass Filter – From Time Domain to Transfer Function



$$\frac{dv_{out}(t)}{dt} = \frac{1}{RC}\,[v_{in}(t) - v_{out}(t)]$$

$$\mathcal{L}\left\{\frac{\mathrm{d}}{\mathrm{dt}}\mathrm{x(t)}\right\} = \mathrm{sX(s)} \qquad \mathcal{L}\{\mathrm{x(t)}\} = \mathrm{X(s)}$$

Laplace properties s. 23

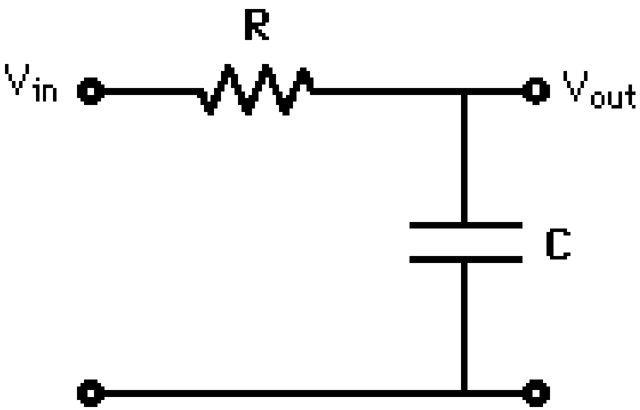
$$sV_{out}(s) = \frac{1}{RC}[(V_{in}(s) - V_{out}(s)]$$

$$V_{out}(s)(1 + sRC) = V_{in}(s)$$

$$1 + sRC = \frac{V_{in}(s)}{V_{out}(s)} = \frac{1}{H(s)}$$

$$\boxed{H(s) = \frac{1}{1 + sRC}}$$

Analog Low-Pass Filter – From Transfer Function to Frequency Response



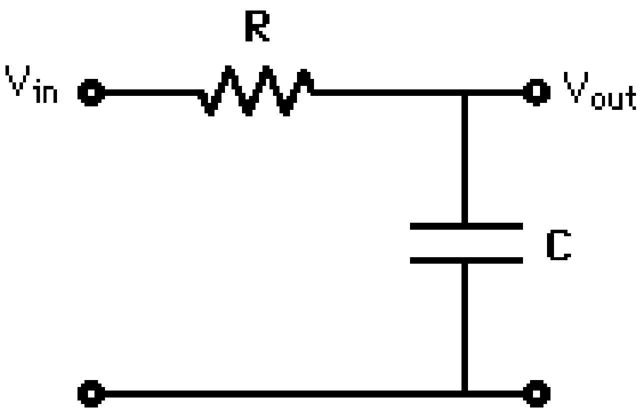
$$H(s) = \frac{1}{1 + sRC}$$

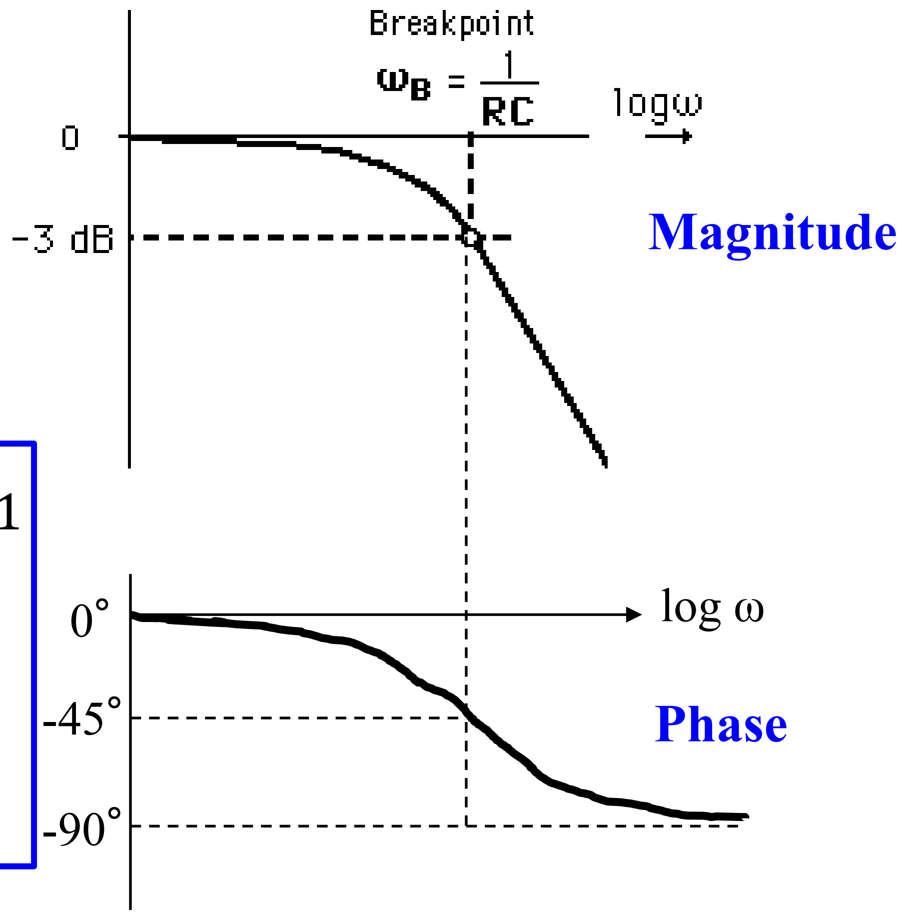
$$H(s = i\omega) = H(\omega) = \frac{1}{1 + i\omega RC}$$

Magnitude: $$|H(\omega)| = \left|\frac{1}{1 + i\omega RC}\right| = \frac{1}{|1 + i\omega RC|} = \frac{1}{\sqrt{1 + \omega^2 R^2 C^2}}$$

Phase: $$\angle H(\omega) = \tan^{-1}\left(\frac{Im[H(\omega)]}{Re[H(\omega)]}\right) = \tan^{-1}(-\omega RC)$$

Analog Low-Pass Filter - Frequency Response (Sketch)

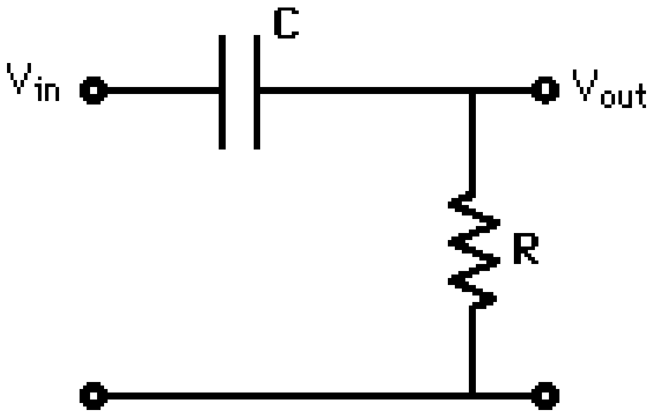




$$|H(\omega_B)| = \frac{1}{\sqrt{1+\omega_B^2 R^2 C^2}} = \frac{1}{\sqrt{2}} = 0.7071$$

i.e. -3 dB = about 71% of the undamped magnitude

Analog High-Pass Filter – From Transfer Function to Frequency Response



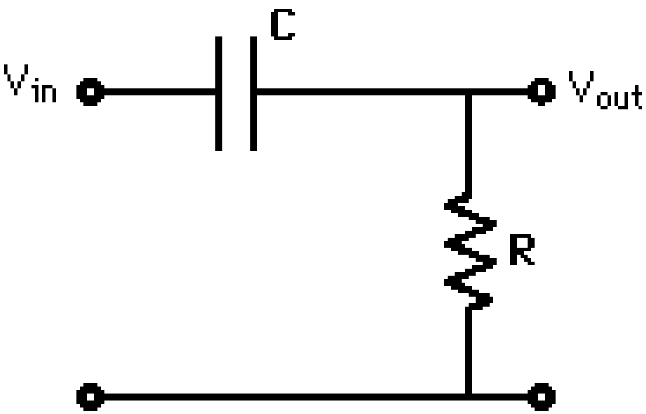
$$H(s) = \frac{sRC}{1+sRC}$$

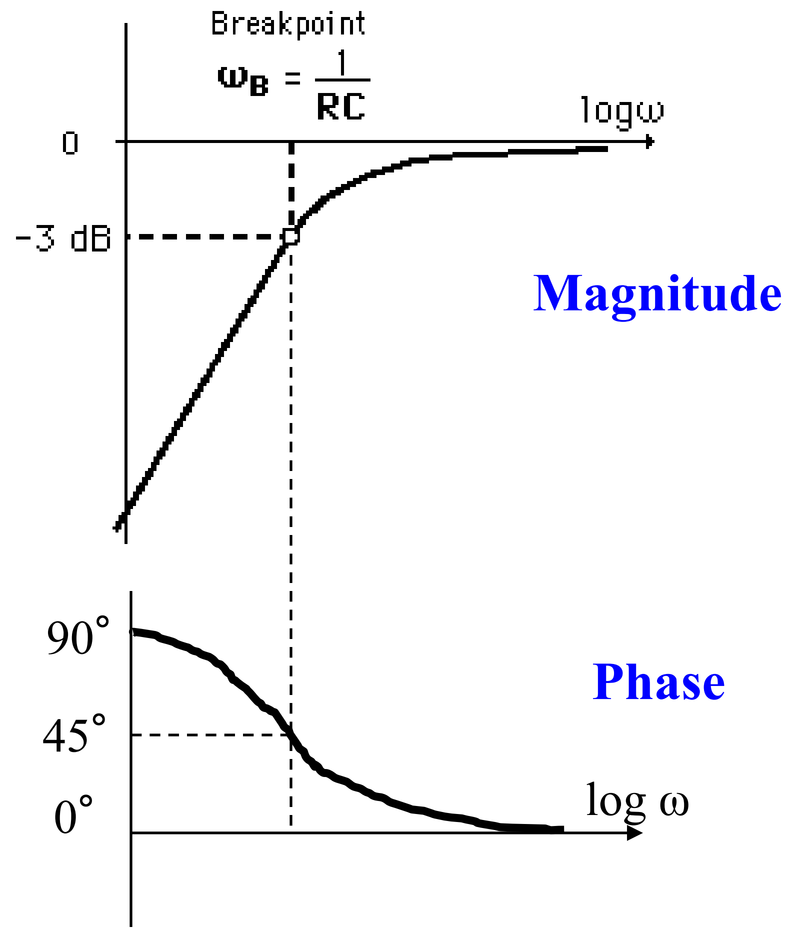
$$H(s=i\omega) = H(\omega) = \frac{i\omega RC}{1+i\omega RC}$$

Magnitude: $$|H(\omega)| = \left|\frac{i\omega RC}{1+i\omega RC}\right| = \frac{|i\omega RC|}{|1+i\omega RC|} = \frac{\omega RC}{\sqrt{1+\omega^2R^2C^2}}$$

Phase: $$\angle H(\omega) = \tan^{-1}\left(\frac{Im[H(\omega)]}{Re[H(\omega)]}\right) = \tan^{-1}\left(\frac{1}{\omega RC}\right)$$

Analog High-Pass Filter - Frequency Response (Sketch)





Conclusion

Take-Home Messages

- The Laplace transform allows for covering the large class of unstable systems
- The continuous-time Fourier transform is a special case of the Laplace transform
- Filters allow a number of operations (e.g., noise removal, anti-aliasing, signal reconstructions, etc.)
- Filters' response can be represented in time (impulse response and step response) and frequency domain (frequency response with Fourier transform, and transfer function with Laplace transform)
- Filters are often easier to design and analyze in the frequency domain

Additional Literature – Week 4

Books

- J. H. McClellan, R. W. Schafer, M. A. Yoder "DSP First: A Multimedia Approach", Prentice Hall, 1999.
- A. Oppenheim and A. S. Willsky with S. Nawab, "Signals and Systems", Prentice Hall, 1997.