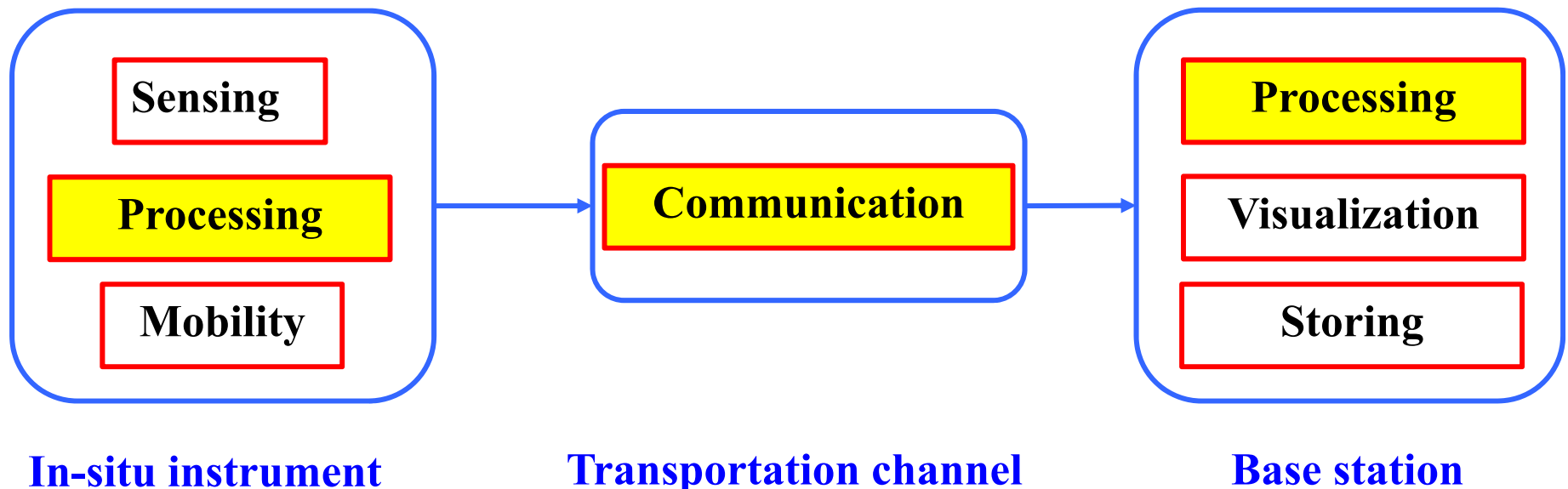


Signals, Instruments, and Systems – W3

Introduction to Signal Processing – Sampling, Reconstruction, and Aliasing

Motivation from Week 1 Lecture

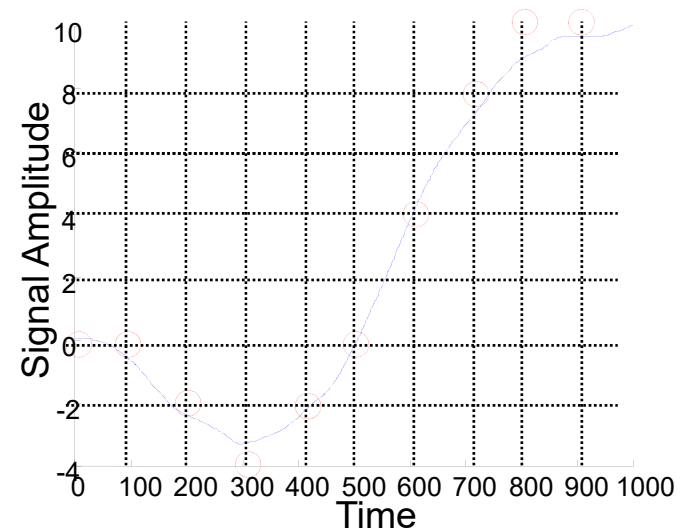


Highlighted blocks are those mainly leveraging the content of this lecture.

Sampling

Analog-Digital Converter (ADC)

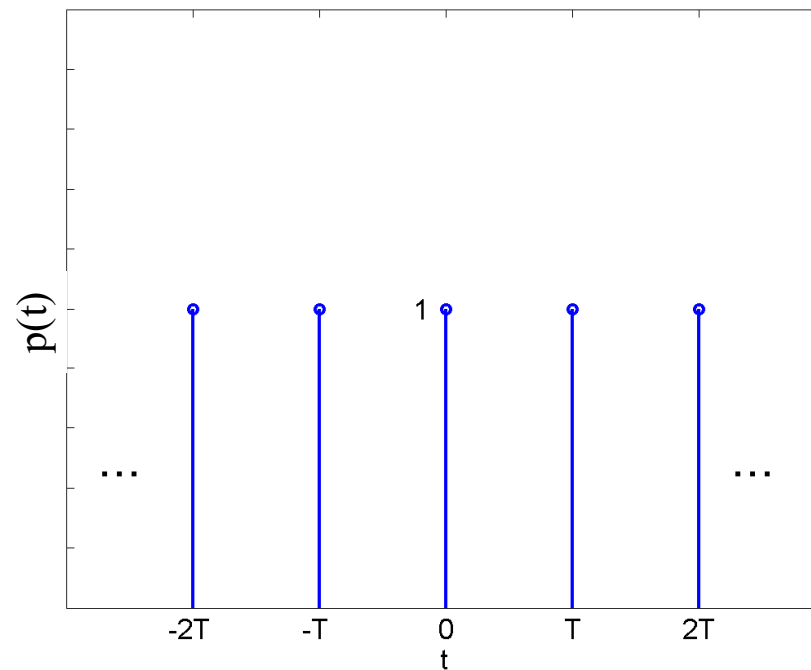
- Transforms continuous analog signal into series of values
- Two key elements
 - **Sampling** (in time)
 - **Quantization** (of values)



$y[n] = 0 \ 0 \ -2 \ -4 \ -2 \ 0 \ 4 \ 8 \ 10 \ 10$

Periodic Train of Dirac Impulses

$$p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$$



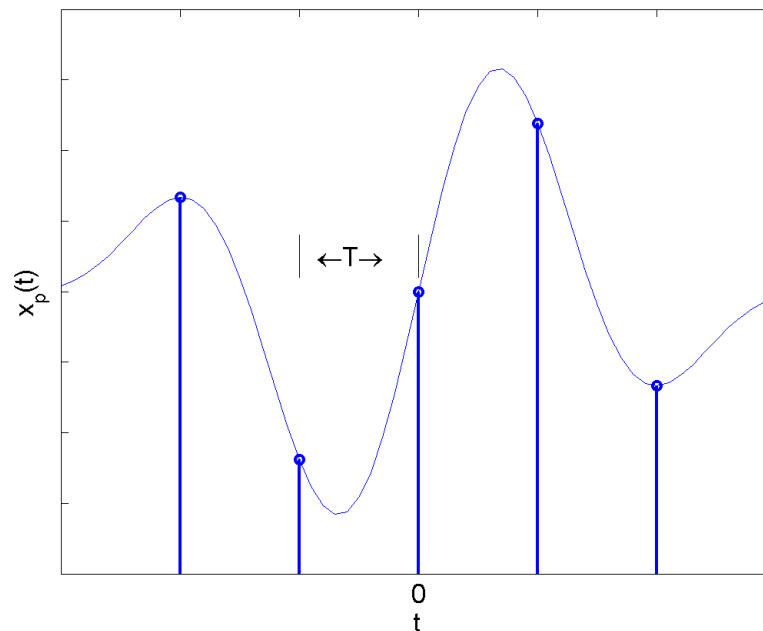
Sampling in Time Domain

$x(t)$

$$p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

Periodic train of Dirac impulses

$$x_p(t) = x(t)p(t) = \sum_{n=-\infty}^{\infty} x(t)\delta(t - nT) = \sum_{n=-\infty}^{\infty} x(nT)\delta(t - nT)$$



T = sampling period
 $f_s = 1/T$ = sampling frequency

How Does Look like the Sampling Operation in the Frequency domain?

- It would be nice to understand how a **sampling** in time process translates in the **frequency domain** so that we can leverage this representation for further reasoning
- Multiplication in time domain means convolution in frequency domain but ...
- How does look like **a periodic train of Dirac impulses in the frequency domain?**

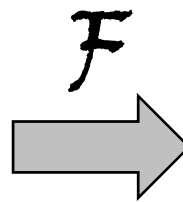
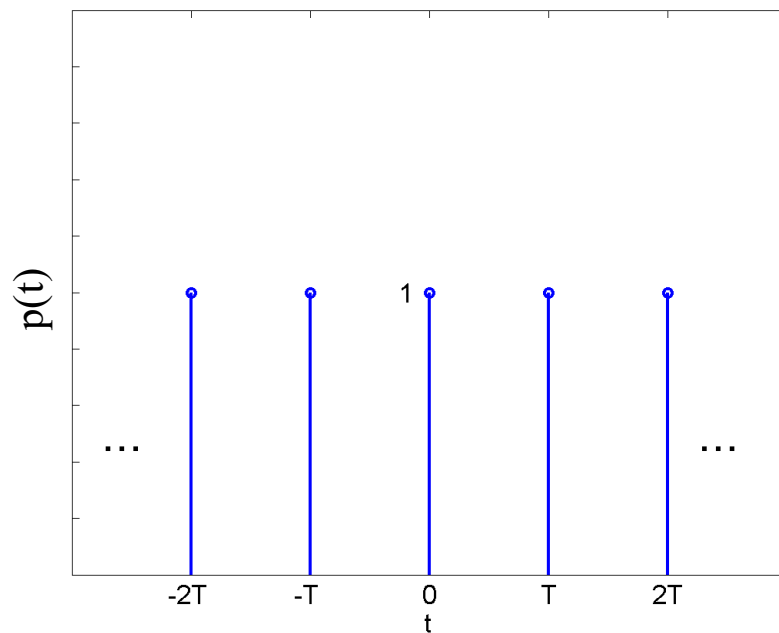
Train of Dirac Impulses in the Frequency Domain?

$$p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

?

Time domain

Frequency domain



?

In order to find out we need to
a detour via Fourier Series!

Complex Coefficients for Arbitrary Period

From W2
(s.13)

Fourier series

$$f(t) = \sum_{n=-\infty}^{\infty} C_n e^{in\omega_0 t}$$

Fourier coefficients
for T periodic function

$$C_n = \frac{1}{T} \int_{-T/2}^{T/2} f(t) e^{-in\omega_0 t} dt$$

Rem: $C_n = |C_n| e^{i\varphi}$

Magnitude: $|C_n|$

Phase: φ

Fourier Transform

Non-unitary, angular frequency notation

From W2
(s.21)

$$F(\omega) = \int_{-\infty}^{+\infty} f(t) e^{-i\omega t} dt$$

**Fourier
Transform**

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} F(\omega) e^{i\omega t} d\omega$$

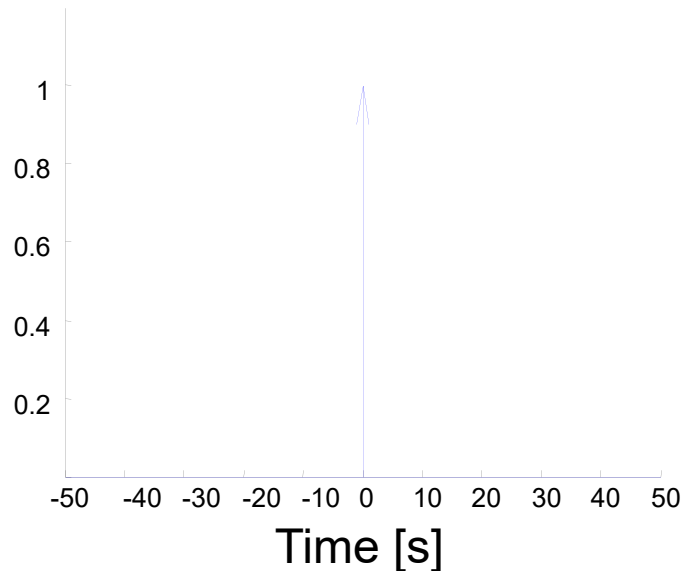
**Inverse Fourier
Transform**

Notes:

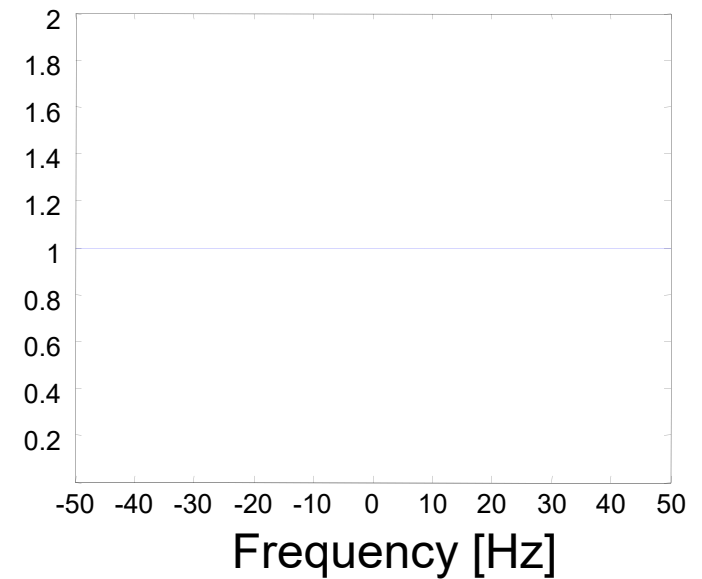
- $\omega = 2\pi\xi \rightarrow$ obtained from unitary, ordinary frequency transform with $\xi = \omega/2\pi$
- F can be replaced with $\hat{f}$
- In electrical engineering i is substituted by j (“i” booked for current)
- Often, in order to emphasize the frequency response aspect, the imaginary aspect of the transform is emphasized: $F(\omega)$, $F(i\omega)$, or $F(j\omega)$ are all equivalent notations

FT of a Dirac Delta Function (or Dirac Impulse)

From W2
(s.23)



$$f(t) = \delta(t)$$



$$\hat{f}(\xi) = 1$$

FT for Periodic Signals

- Although it generalizes to aperiodic signals, the FT can be also applied to periodic signals
- We can derive the FT of a periodic signal from its Fourier series (which have been developed for periodic signals, see W2)
- The FT of a periodic Dirac train of impulses consists of a **periodic train of impulses in the frequency domain** as well, with the area of the impulses proportional to the Fourier series coefficients

FT for Periodic Signals

(assume: period T , $\omega_0 = 2\pi/T$)

$$X(\omega) = 2\pi\delta(\omega - \omega_0)$$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} 2\pi\delta(\omega - \omega_0) e^{i\omega t} d\omega$$

W2, s. 21, IFT

$$x(t) = e^{i\omega_0 t}$$

See also W2, s. 28,
modulation property

Linear combination of impulses equally spaced in frequency:

$$X(\omega) = \sum_{n=-\infty}^{+\infty} 2\pi C_n \delta(\omega - n\omega_0)$$

W2, s. 28, linearity
property

$$x(t) = \sum_{n=-\infty}^{+\infty} C_n e^{in\omega_0 t}$$

Compare with
W2, s. 13, Fourier series

(assume: period T , $\omega_0 = 2\pi/T$)

Now consider the periodic impulse train of before:

$$p(t) = \sum_{n=-\infty}^{+\infty} \delta(t - nT)$$

Calculate the coefficient of its Fourier series:

$$C_n = \frac{1}{T} \int_{-T/2}^{+T/2} \delta(t) e^{-in\omega_0 t} dt = \frac{1}{T}$$

W2, s. 13, Fourier series coefficients

This means that each Fourier coefficient of the periodic impulse train has the same value; insert C_n in previous expression (s. 13):

$$P(\omega) = \sum_{n=-\infty}^{+\infty} 2\pi C_n \delta(\omega - n\omega_0)$$

$$P(\omega) = \sum_{n=-\infty}^{+\infty} \frac{2\pi}{T} \delta(\omega - n\omega_0) = \frac{2\pi}{T} \sum_{n=-\infty}^{+\infty} \delta(\omega - n\omega_0)$$

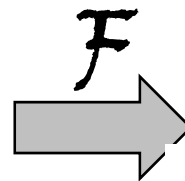
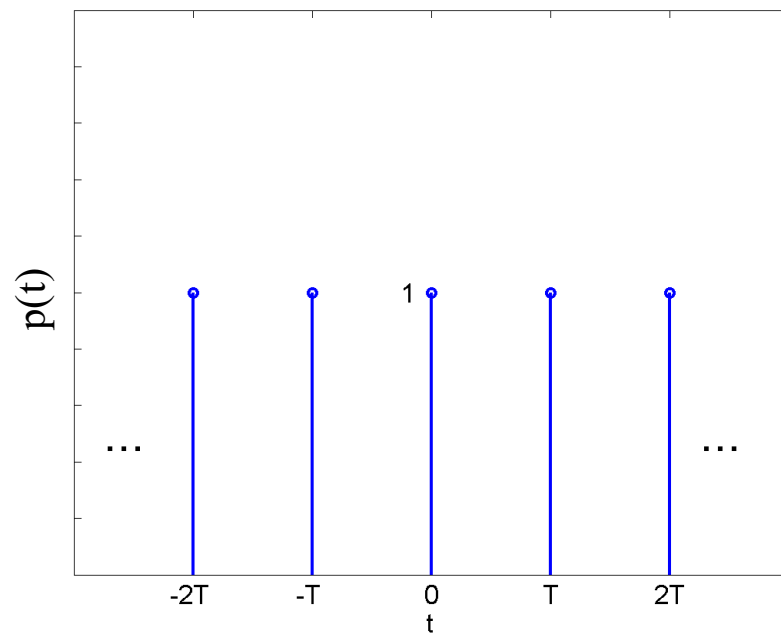
Train of Dirac Impulses

$$\omega_s = \frac{2\pi}{T} \quad \text{Sampling angular frequency}$$

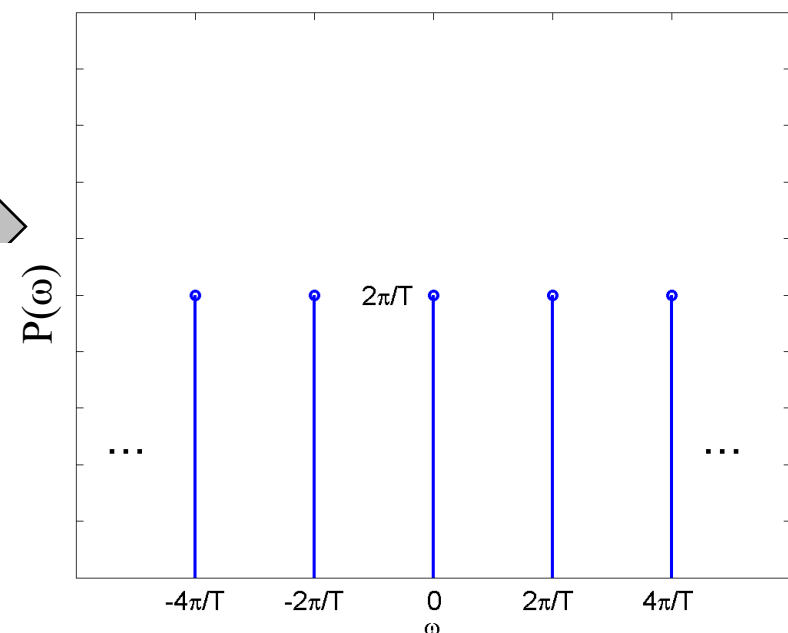
$$p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

$$P(\omega) = \sum_{k=-\infty}^{\infty} \frac{2\pi}{T} \delta(\omega - k\omega_s)$$

Time domain



Frequency domain



Sampling in Frequency Domain

Time domain

$$x_p(t) = x(t)p(t)$$

Frequency domain

$$X_p(\omega) = \frac{1}{2\pi} X(\omega) * P(\omega)$$

FT properties

$$P(\omega) = \frac{2\pi}{T} \sum_{k=-\infty}^{\infty} \delta(\omega - k\omega_s)$$

From s.15

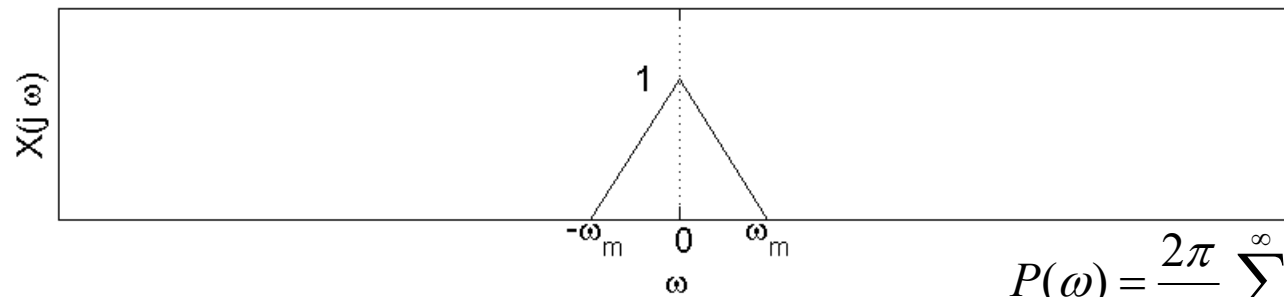
$$X_p(\omega) = \frac{1}{T} \sum_{k=-\infty}^{\infty} X(\omega) * \delta(\omega - k\omega_s)$$

Note: see also W2 ss.
43-45 as examples for
this operation

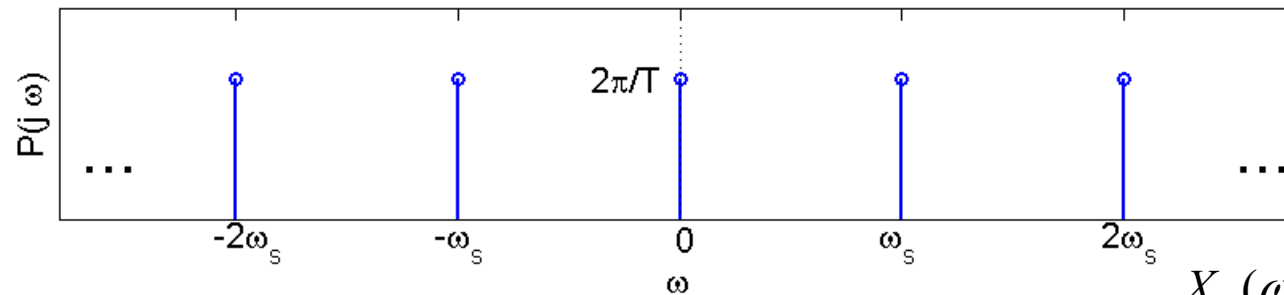
$$= \frac{1}{T} \sum_{k=-\infty}^{\infty} X(\omega - k\omega_s)$$

Sampling a Band-Limited Signal

$X(\omega) \rightarrow$ spectrum of signal $x(t)$
with highest frequency $< \omega_m$

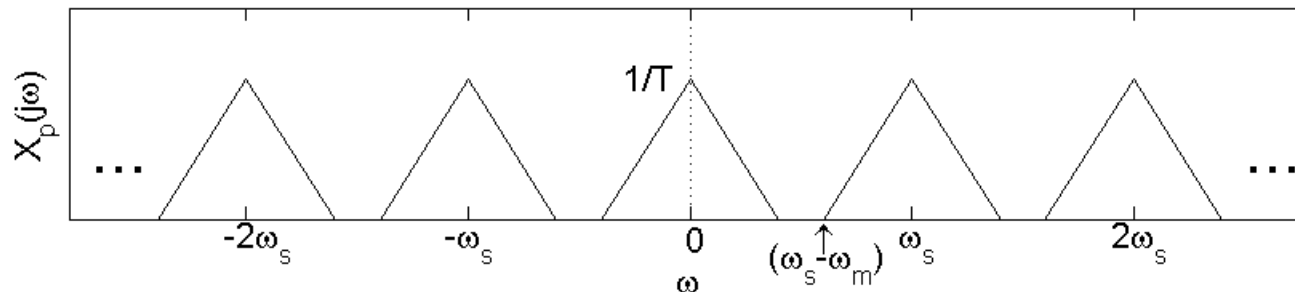


$$P(\omega) = \frac{2\pi}{T} \sum_{k=-\infty}^{\infty} \delta(\omega - k\omega_s)$$



Angular sampling
frequency
 $\omega_s > 2\omega_m$

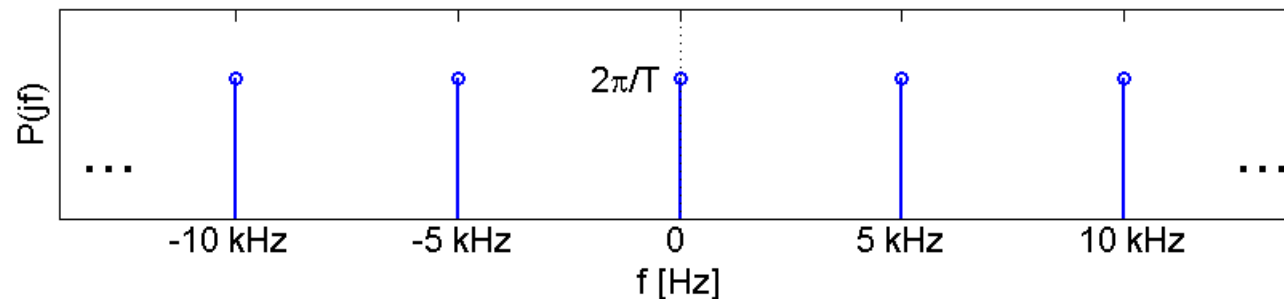
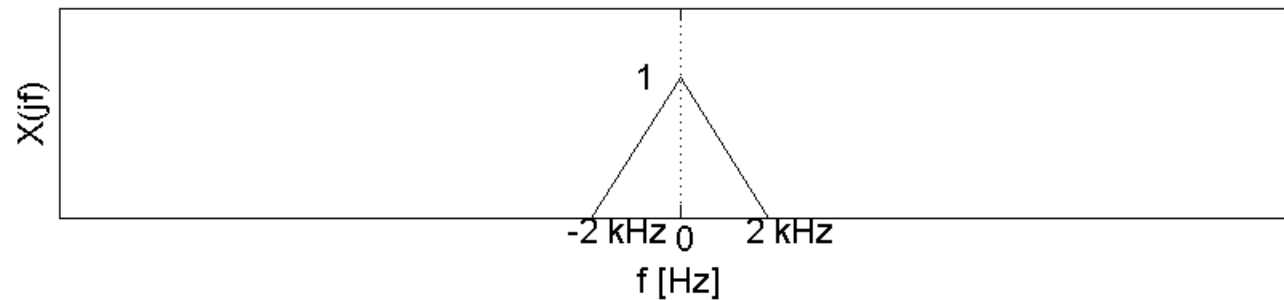
$$X_p(\omega) = \frac{1}{2\pi} X(\omega) * P(\omega)$$



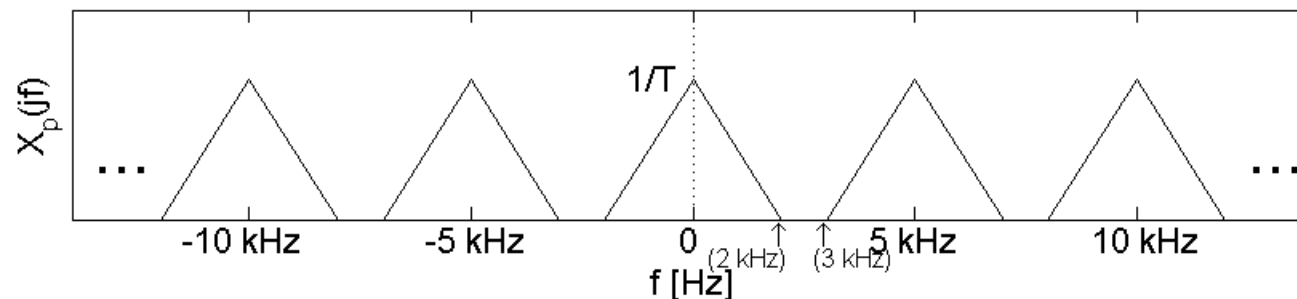
$$= \frac{1}{T} \sum_{k=-\infty}^{\infty} X(\omega - k\omega_s)$$

Sampling a Band-Limited Signal

$X(\xi) = X(f) \rightarrow$ spectrum of signal $x(t)$
with highest frequency < 2 kHz

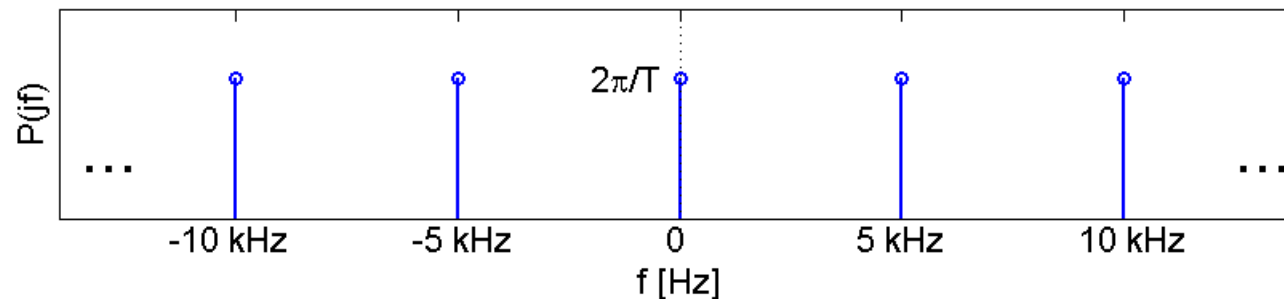
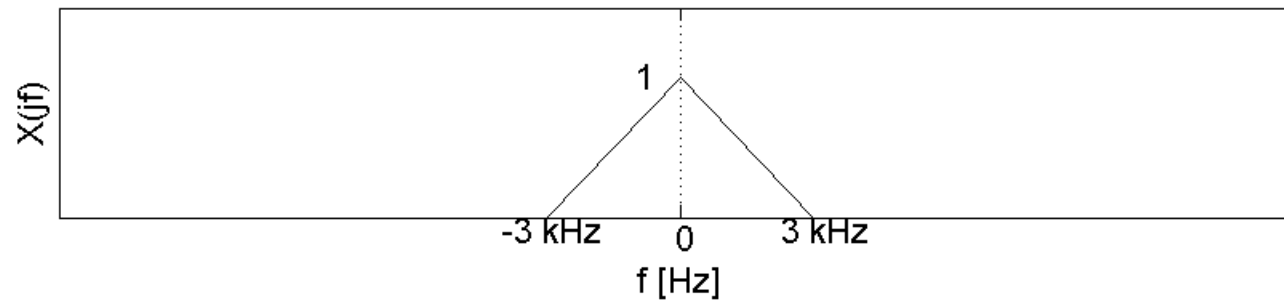


Sampling frequency:
5 kHz $> 2 * 2$ kHz

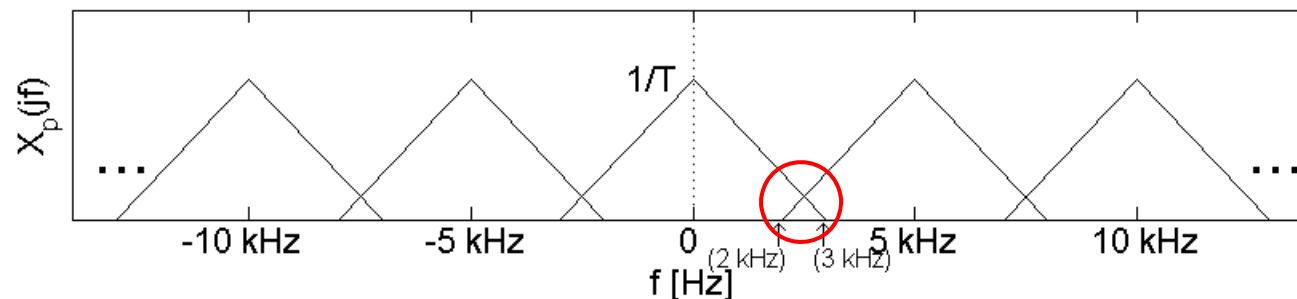


Sampling a Band-Limited Signal

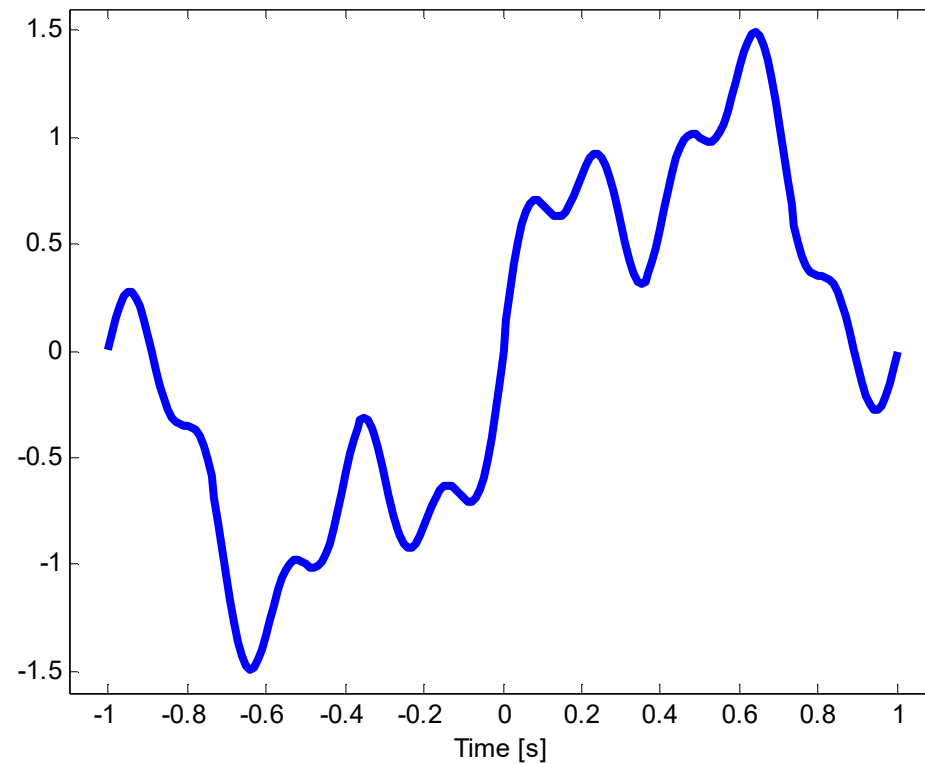
$X(\xi) = X(f) \rightarrow$ spectrum of signal $x(t)$
with highest frequency < 3 kHz



Sampling frequency:
 $5 \text{ kHz} < 2 * 3 \text{ kHz}$

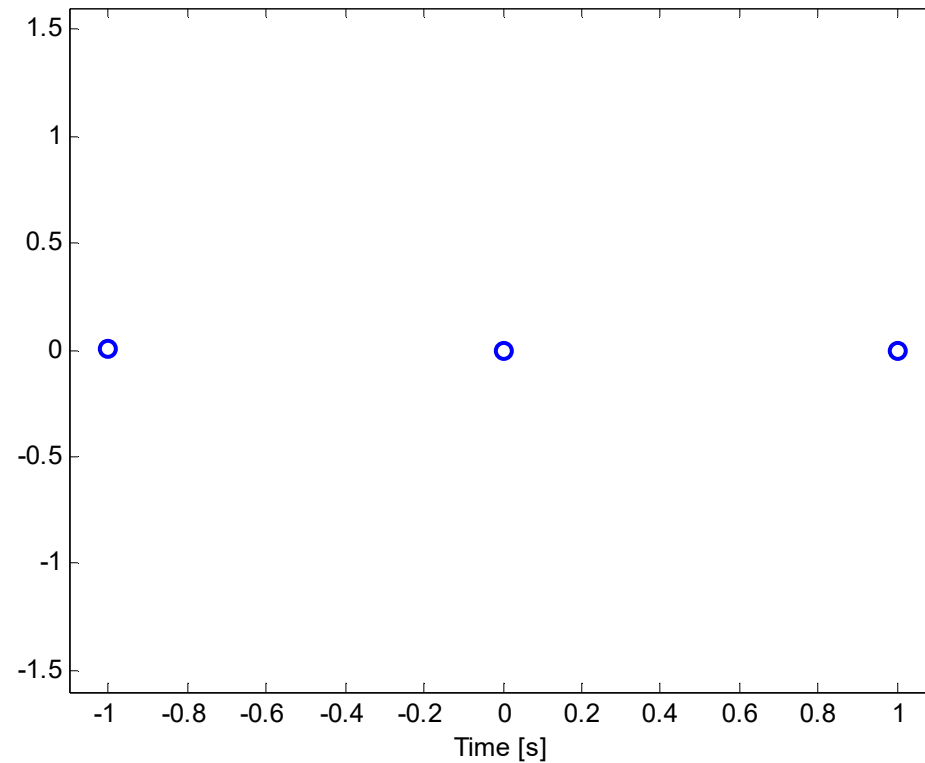


Original Signal



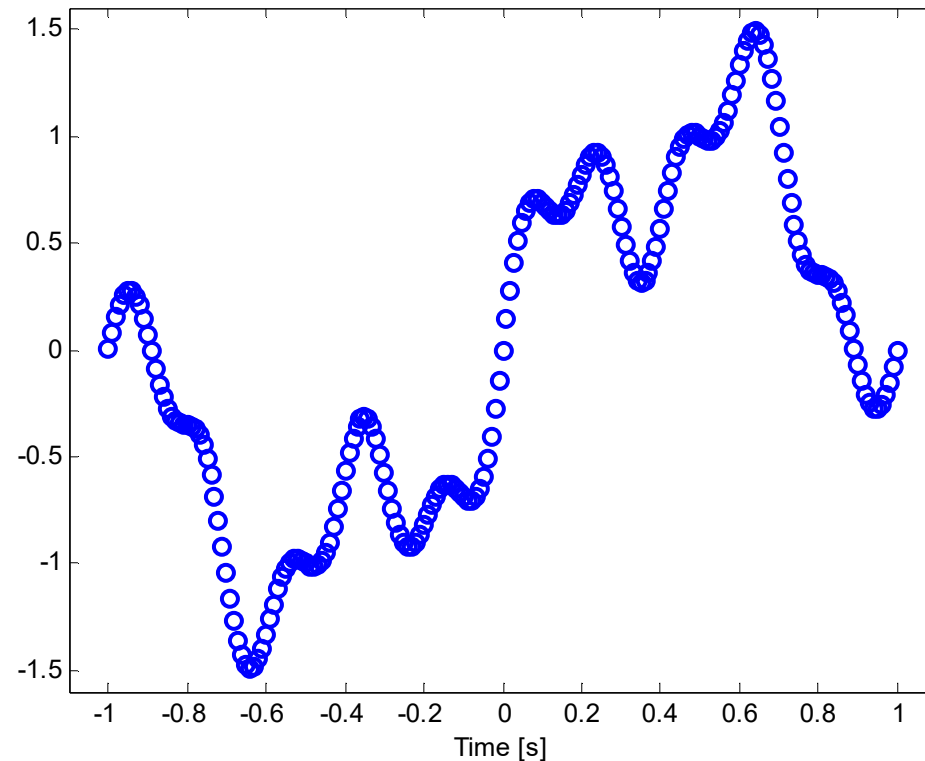
$$f(t) = \sin(2\pi t) + 0.4 \sin(2\pi \cdot 2t) + 0.2 \sin(2\pi \cdot 5t)$$

Too Few Samples (1Hz)



→ Data is lost

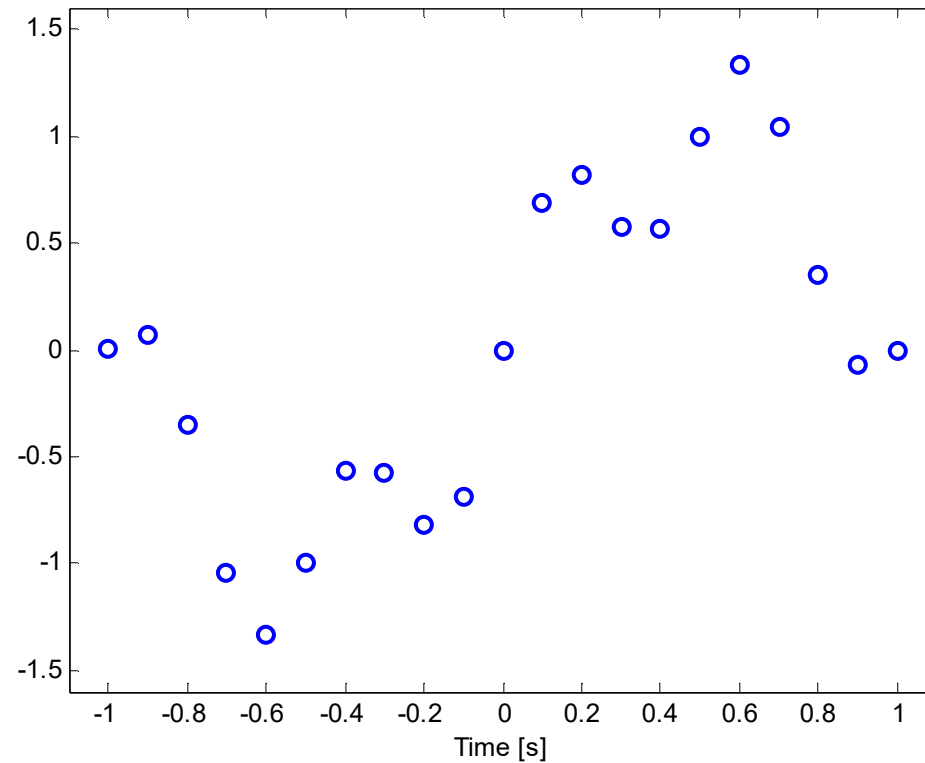
Too Many Samples (100 Hz)



→ Redundant data

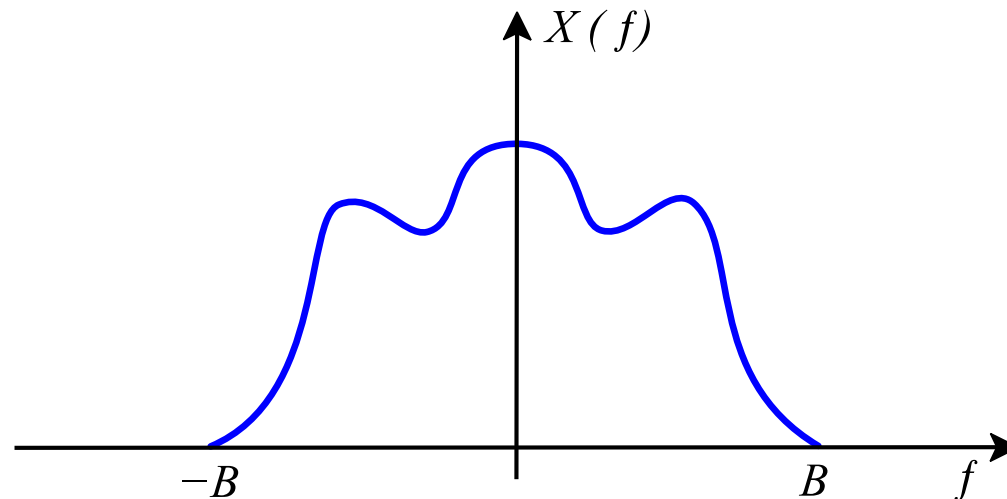
→ Increase of data size

Minimal Possible Sampling (> 10 Hz)



Nyquist–Shannon Theorem

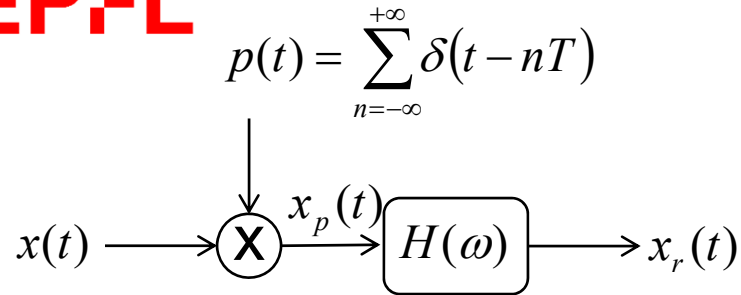
- If a function $x(t)$ contains no frequencies higher than B Hz, it is completely determined by giving its coordinates at a series of points spaced $1/(2B)$ seconds apart.
- Sampling frequency must be at least two times greater than the maximal signal frequency



Sampling in Practice

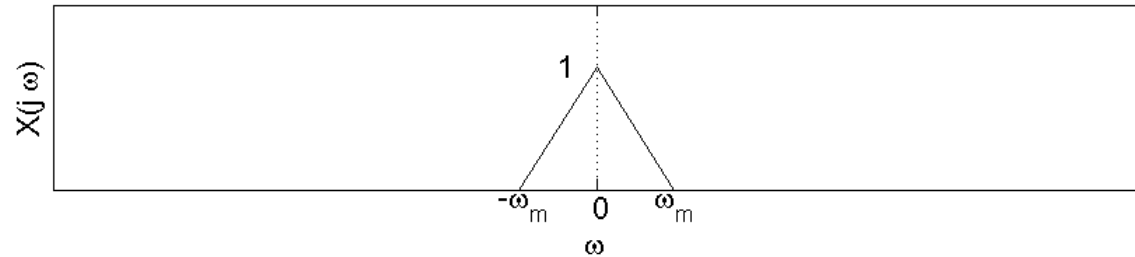
- Sampling frequency two times greater than maximal frequency is the limit
- Example (parsimony principle applied): audio CD, sampling at 44.1 kHz since maximal hearable frequency is 20 kHz (see W1, s. 62)
- If affordable, try to use a sampling frequency 10 times greater than the maximal frequency (help all sorts of filtering and reconstruction processes)

Signal Reconstruction



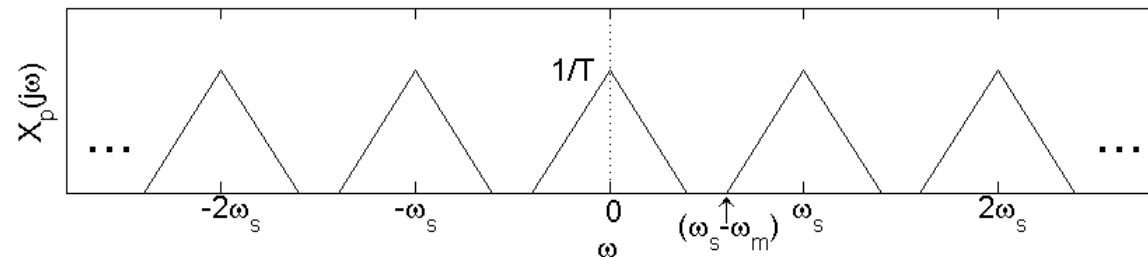
If there is no overlap between the shifted spectra the signal $x_r(t)$ can be perfectly reconstructed from $x(t)$

spectrum of original signal



spectrum of sampled signal

sampling angular frequency
 $\omega_s > 2 \omega_m$

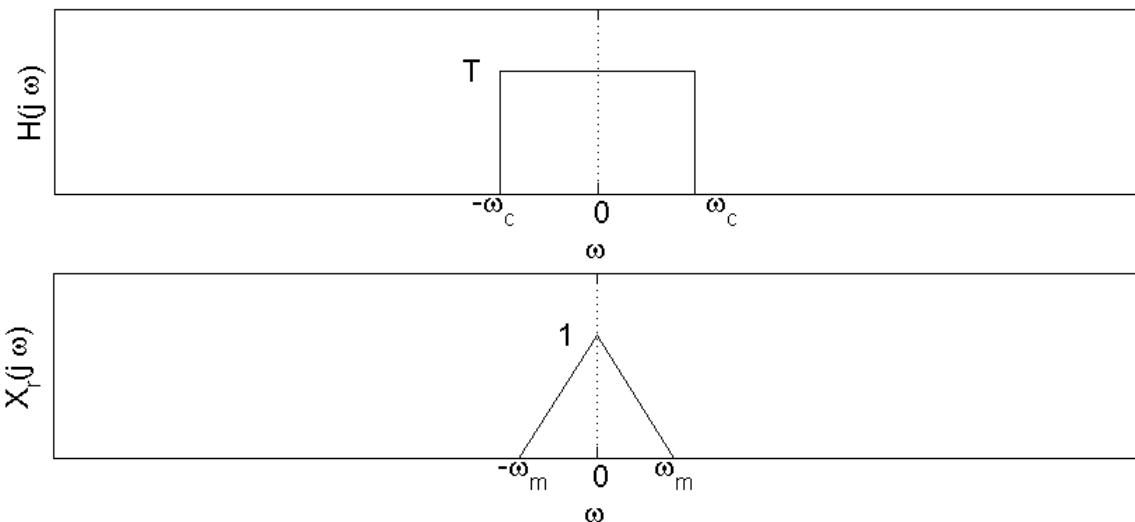


filtering

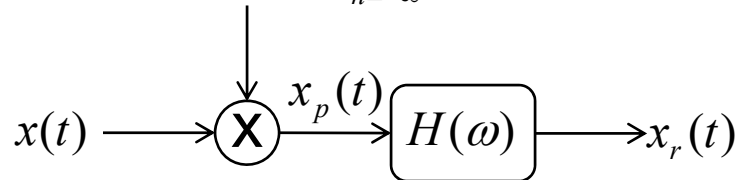
filter cut-off angular frequency
 $\omega_m < \omega_c < (\omega_s - \omega_m)$

$$X_r(\omega) = X_p(\omega)H(\omega)$$

spectrum of reconstructed signal

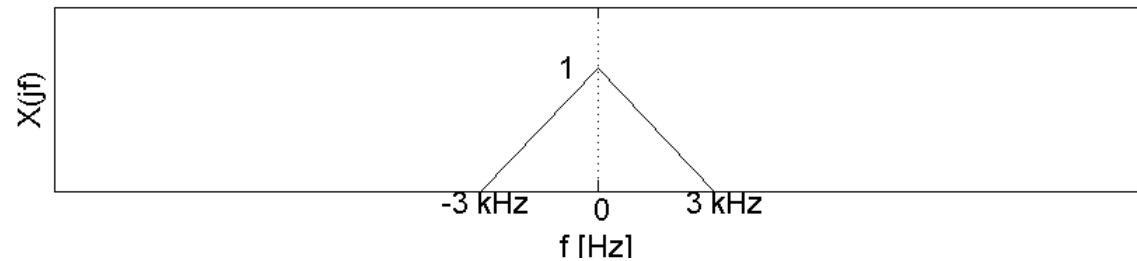


$$p(t) = \sum_{n=-\infty}^{+\infty} \delta(t - nT)$$



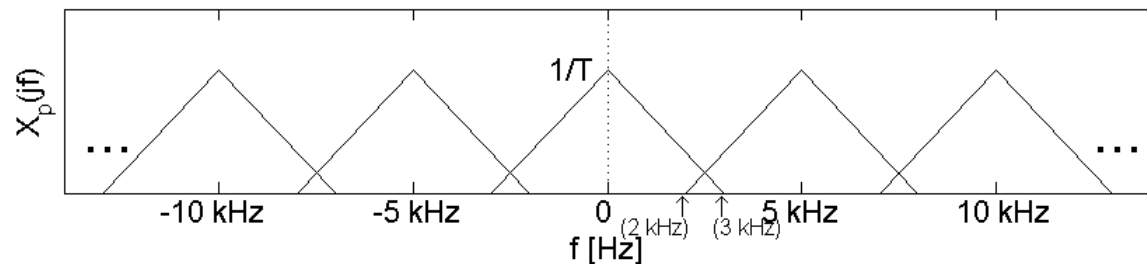
If there is overlap between the shifted spectra the signal $x_r(t)$ cannot be perfectly reconstructed from $x(t)$

spectrum of original signal



spectrum of sampled signal

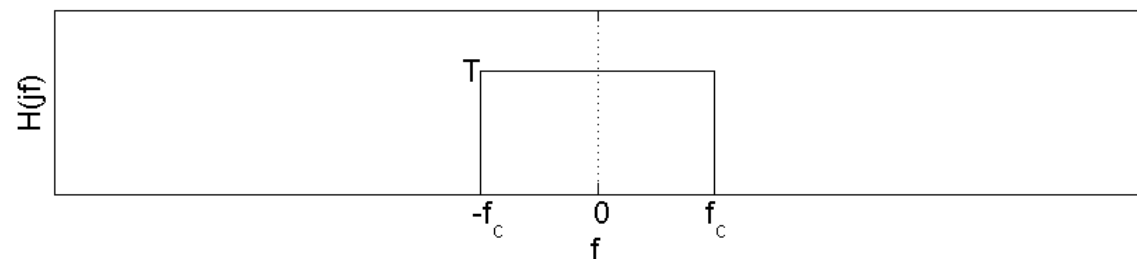
sampling frequency
 $f_s < 2 f_m$



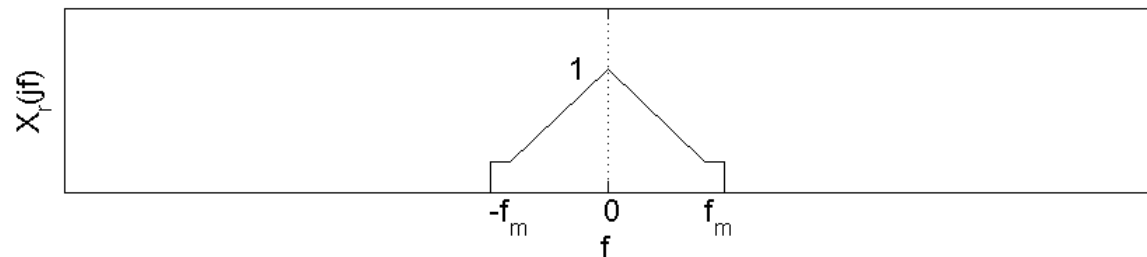
filtering

filter cut-off frequency f_c
 $f_m < f_c < (f_s - f_m)$

$$X_r(\omega) = X_p(\omega)H(\omega)$$

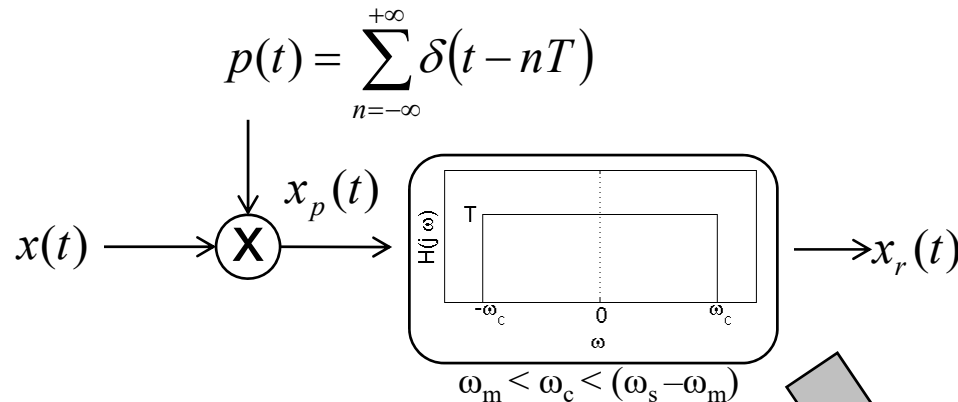


spectrum of reconstructed signal



Time Domain Interpretation of Signal Reconstruction

- Multiplication with a Low-Pass Filter (LPF) in the frequency domain
- The LPF interpolates the samples assuming $x(t)$ contains no energy at frequencies $> \omega_c$ (ω_c = cutoff angular frequency)



$$x_r(t) = x_p(t) * h(t)$$

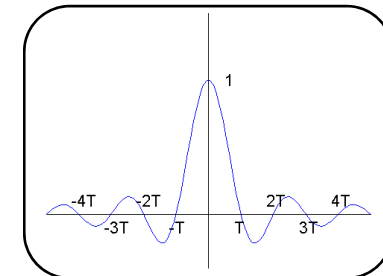
$$= \left(\sum_{n=-\infty}^{\infty} x(nT) \delta(t - nT) \right) * h(t)$$

$$= \sum_{n=-\infty}^{\infty} x(nT) h(t - nT)$$

$$= \sum_{n=-\infty}^{\infty} x(nT) \frac{T \sin(\omega_c(t - nT))}{\pi(t - nT)}$$

with $h(t) = \frac{T \sin(\omega_c t)}{\pi t}$

Note: see also W2 ss. 43-45 as examples for this operation



Signal Reconstruction in Practice

1. Whittaker-Shannon interpolation (band-limited interpolation):

- Signal has to be band limited
(i.e. Fourier transform for frequencies greater than B equal 0)
- The sampling rate must exceed twice the bandwidth, $2B$, i.e. $f_s > 2B$

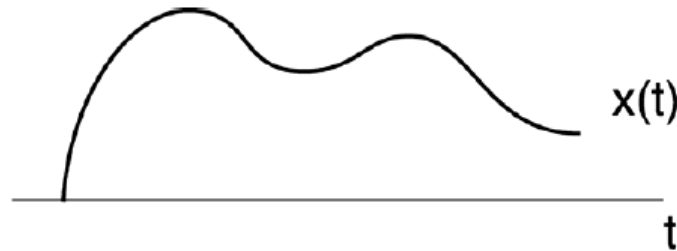
Assume: $\omega_c = \frac{\omega_s}{2} = \frac{2\pi}{2T} = \frac{\pi}{T}$ in s. 29, $\text{sinc}(x) = \frac{\sin(\pi x)}{\pi x}$ Normalized sinc function

$$x_r(t) = \sum_{n=-\infty}^{\infty} x(nT) \cdot \text{sinc}\left(\frac{t - nT}{T}\right)$$

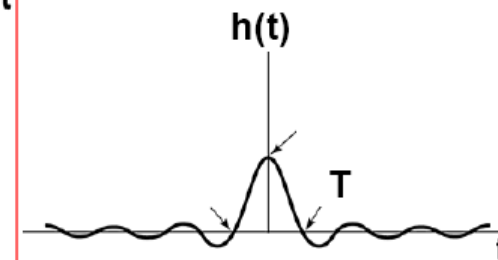
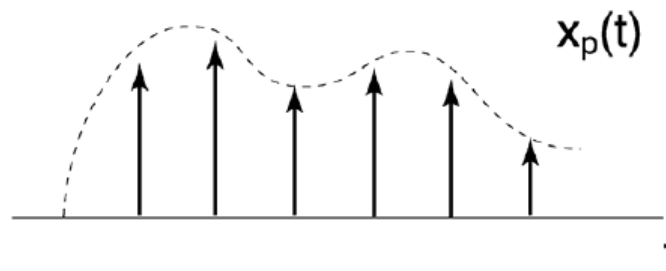
$$x_r(t) = \left(\sum_{n=-\infty}^{\infty} x(nT) \cdot \delta(t - nT) \right) * \text{sinc}\left(\frac{t}{T}\right) \quad (\text{Alternative equivalent formulation})$$

Graphic Illustration of Time-Domain Interpolation

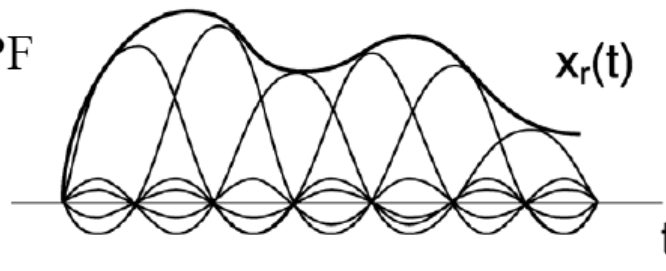
Original
CT signal



After sampling

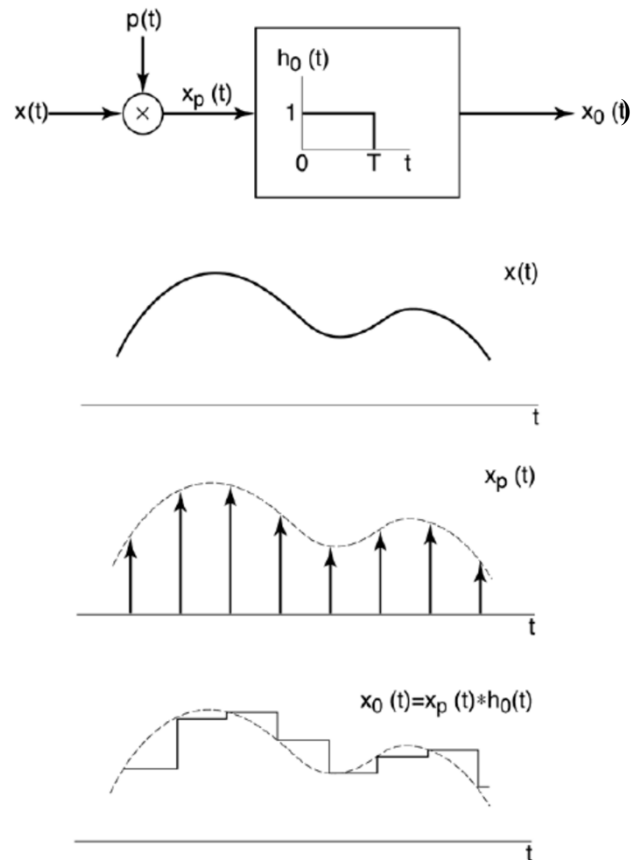


After passing the LPF
(Low Pass Filter)

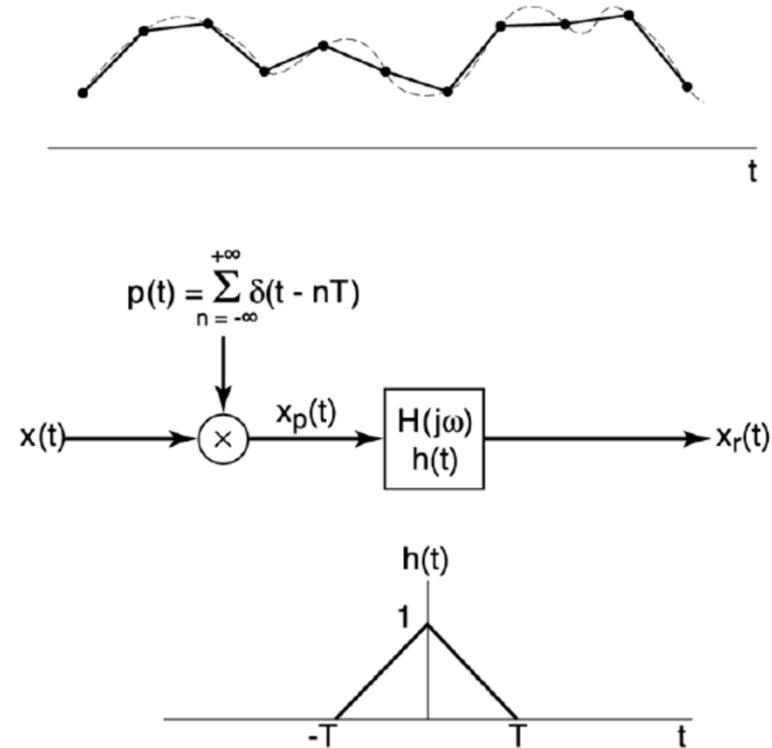


Signal Reconstruction in Practice

2. Zero-order hold

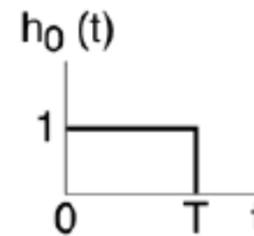


3. First-order hold (linear interpolation)

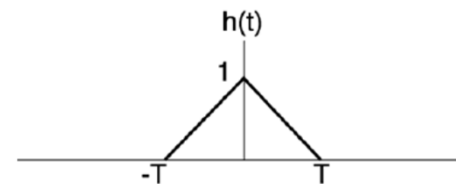


The reconstructed signal $x_r(t)$ is obtained through a **convolution** between the sampled signal $x_p(t)$ with period T and one of the following three **interpolation functions** $h(t)$.

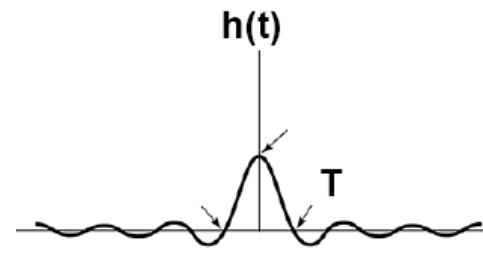
1. Zero-order hold (ZOH)



2. First-order hold (FOH)



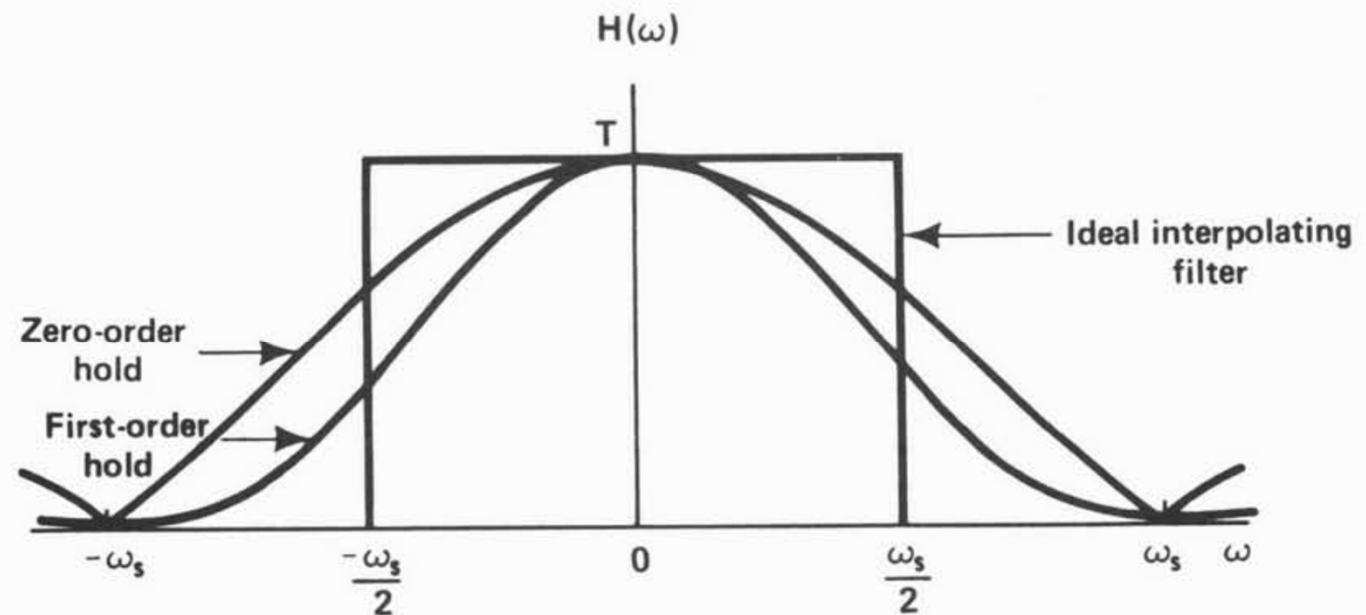
3. Whittaker-Shannon



Computational cost

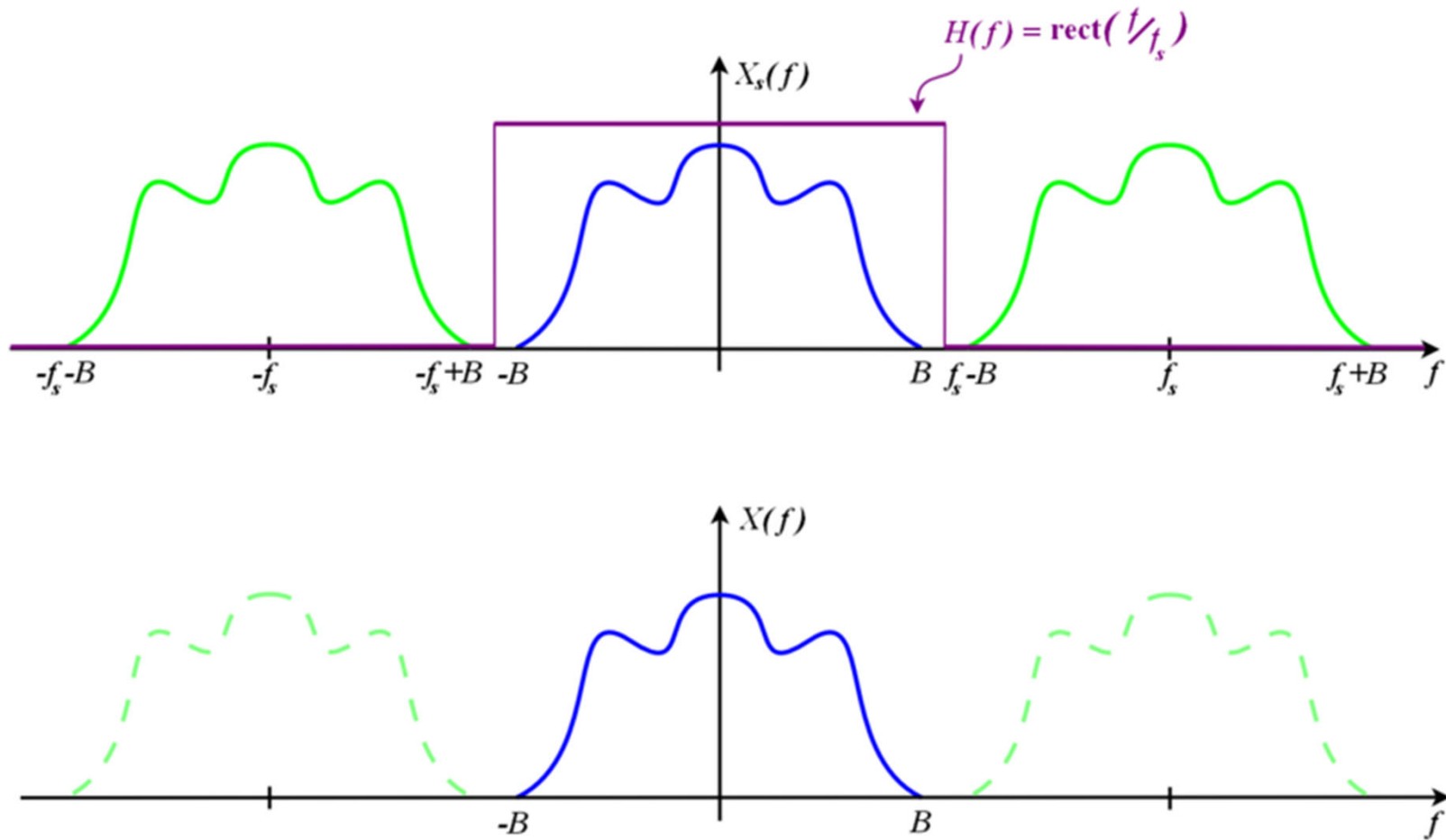
Reconstruction Summary – Frequency Domain

The spectrum of the reconstructed signal $X_r(\omega)$ is obtained through a **multiplication** between the spectrum of the sampled signal $X_p(\omega)$ with angular sampling frequency ω_s and one of the following three **low-pass filters**.



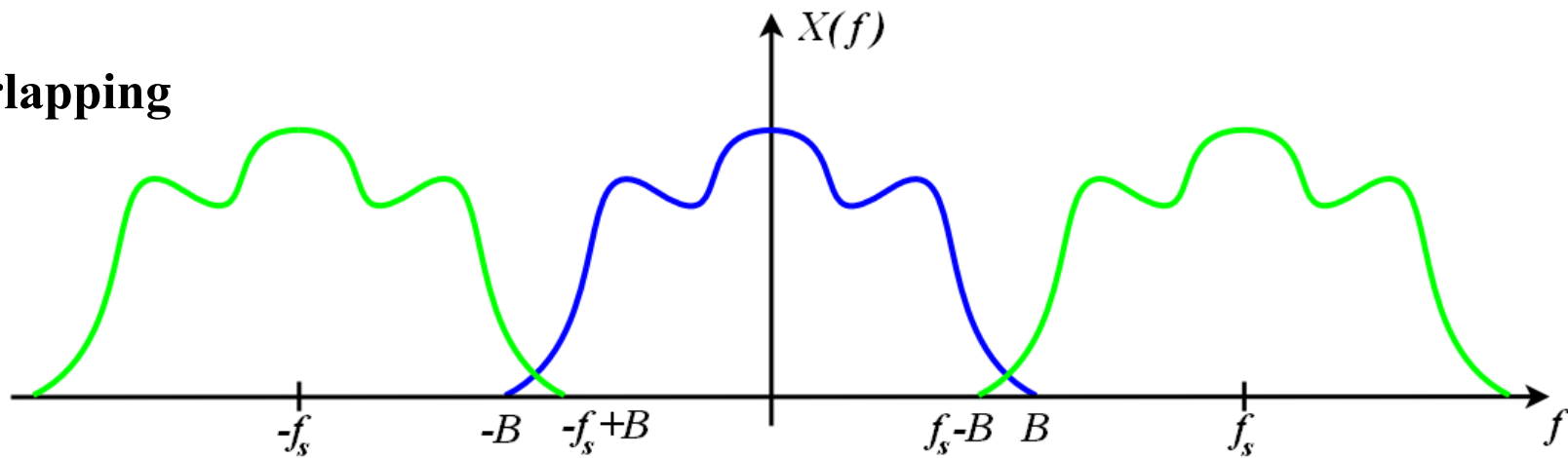
Aliasing

No Problems in Reconstruction

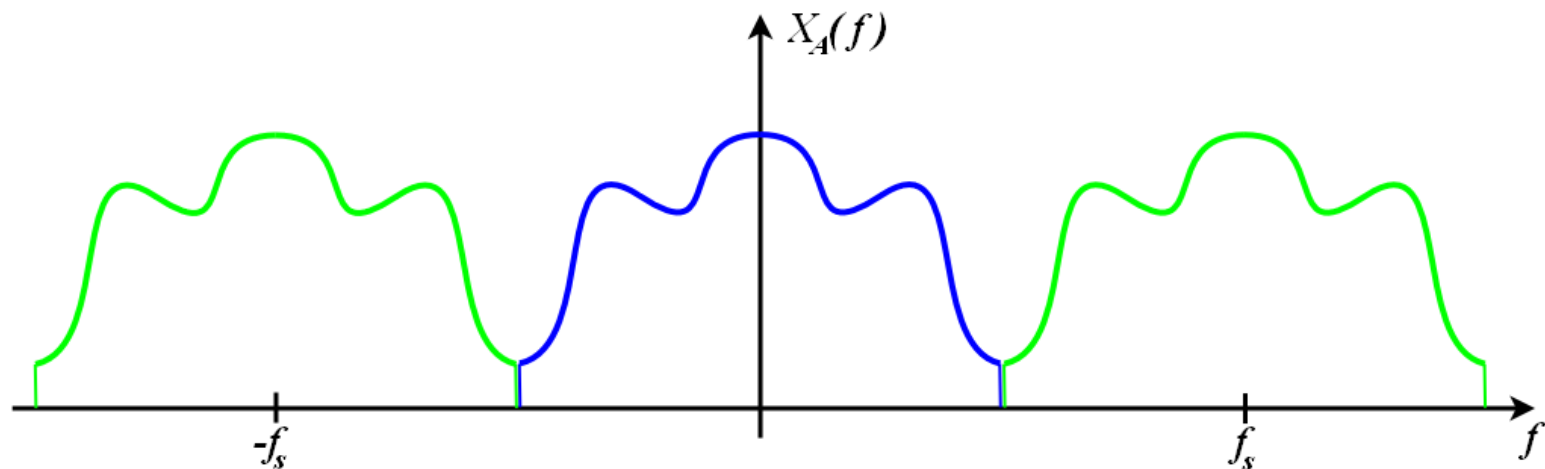


Reconstruction Problems

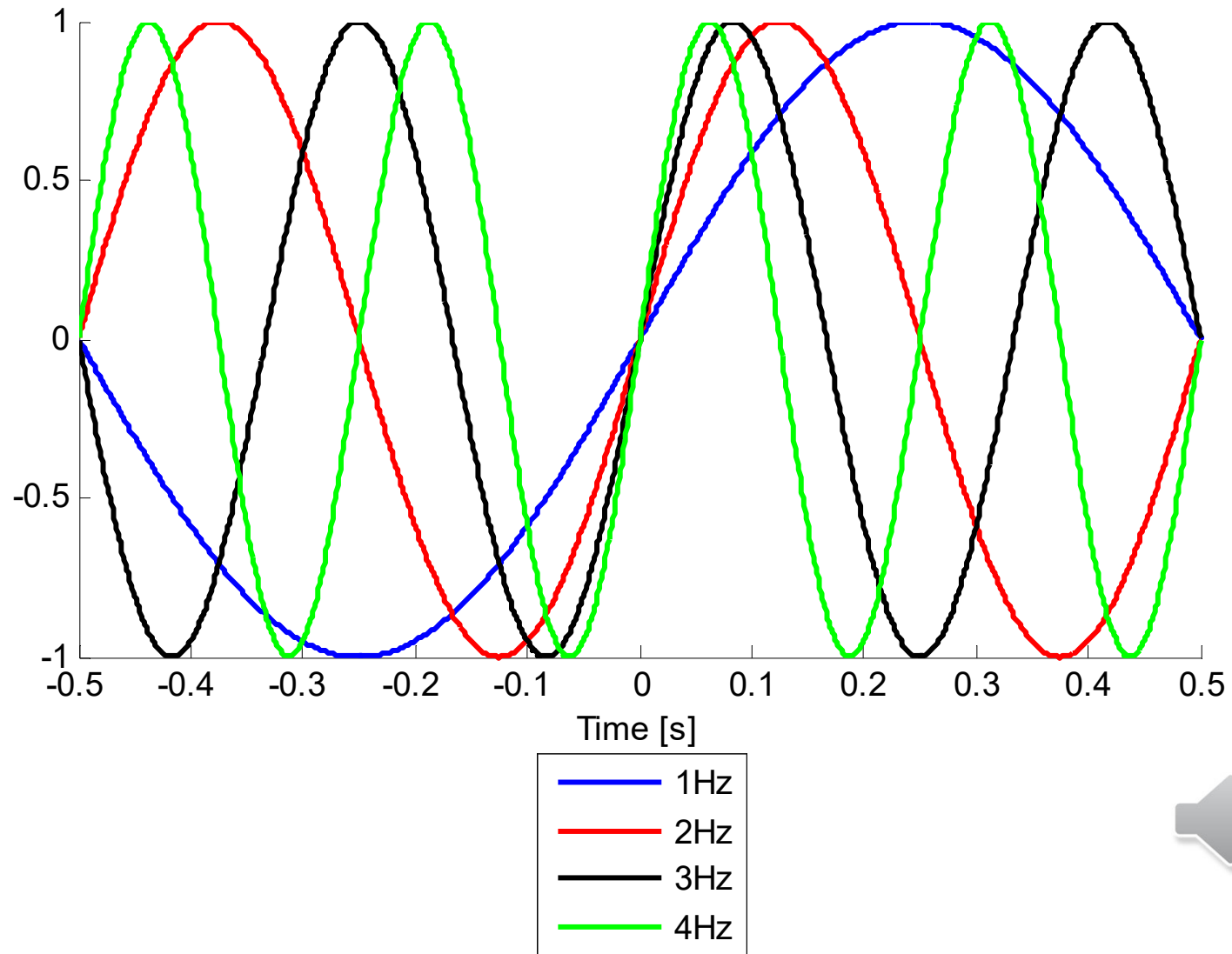
Overlapping



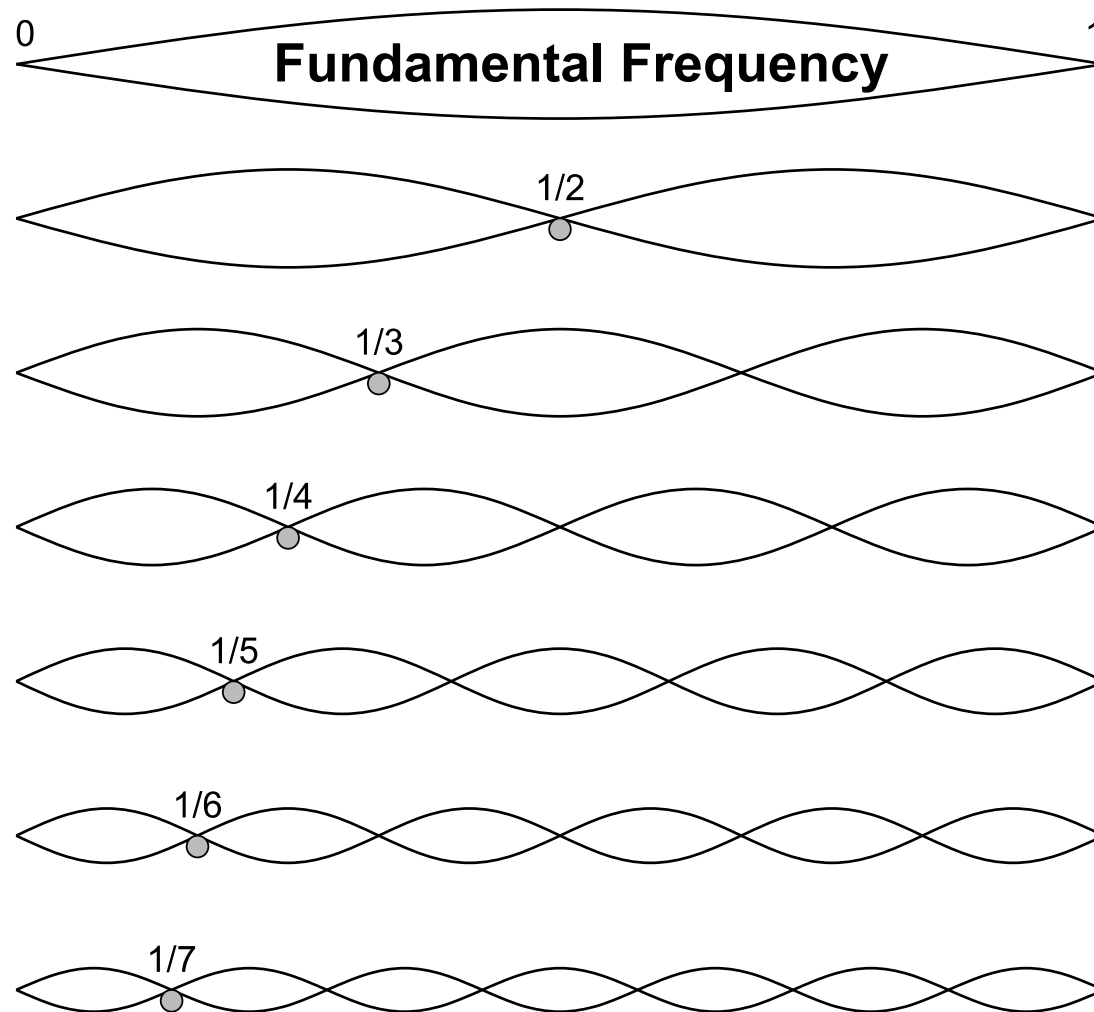
Alias



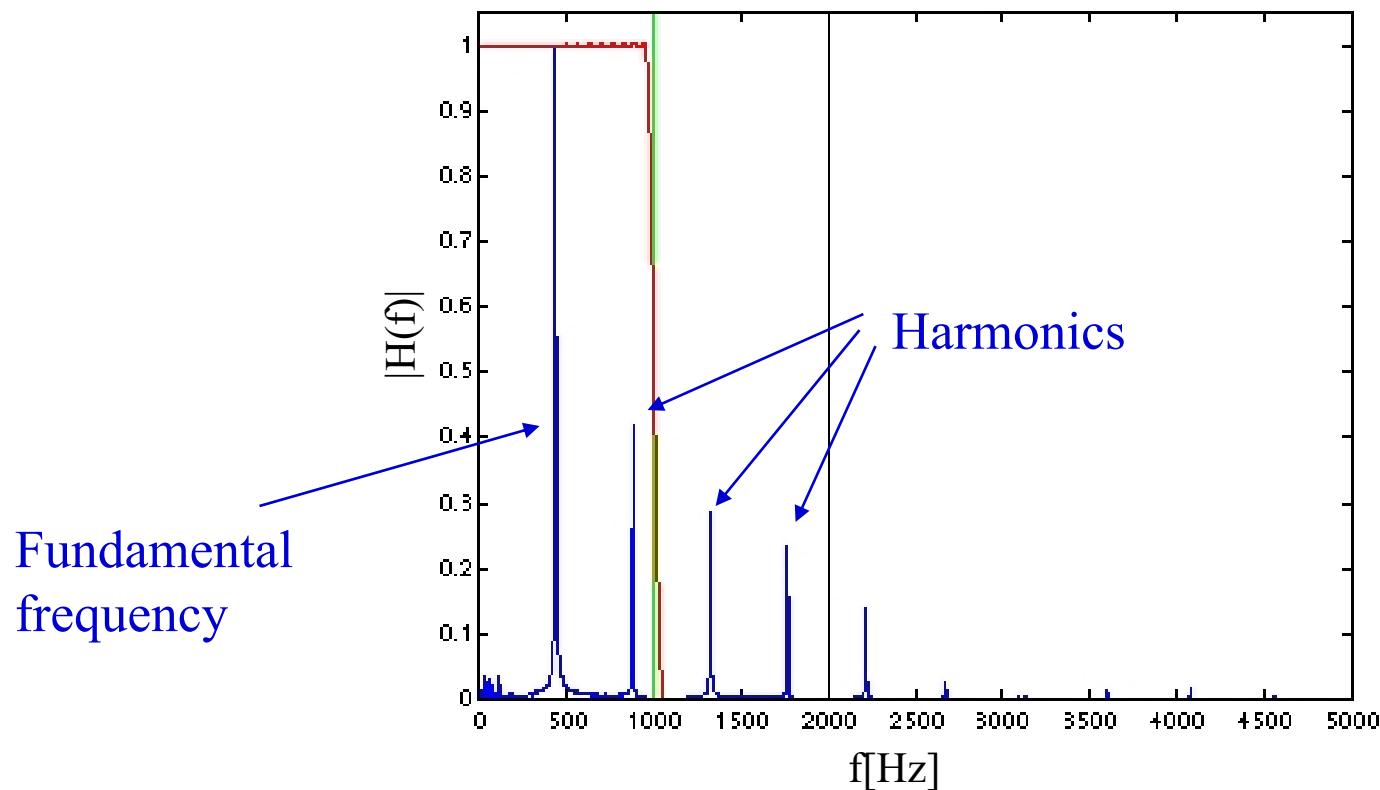
Harmonics



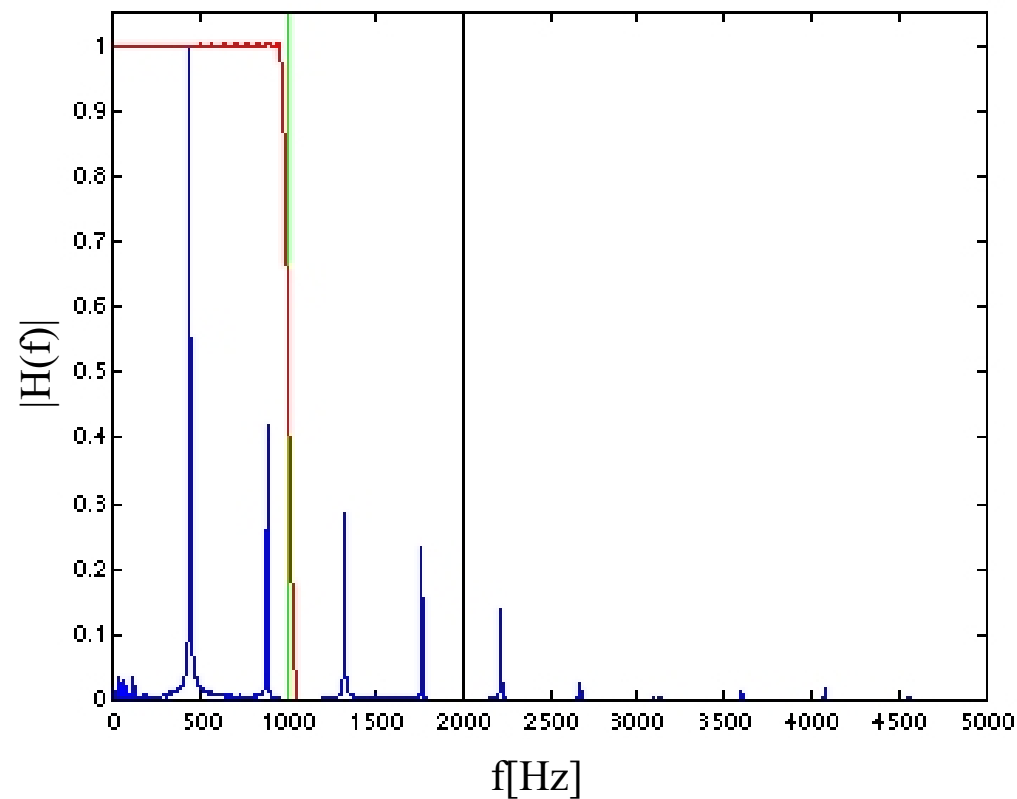
Harmonics



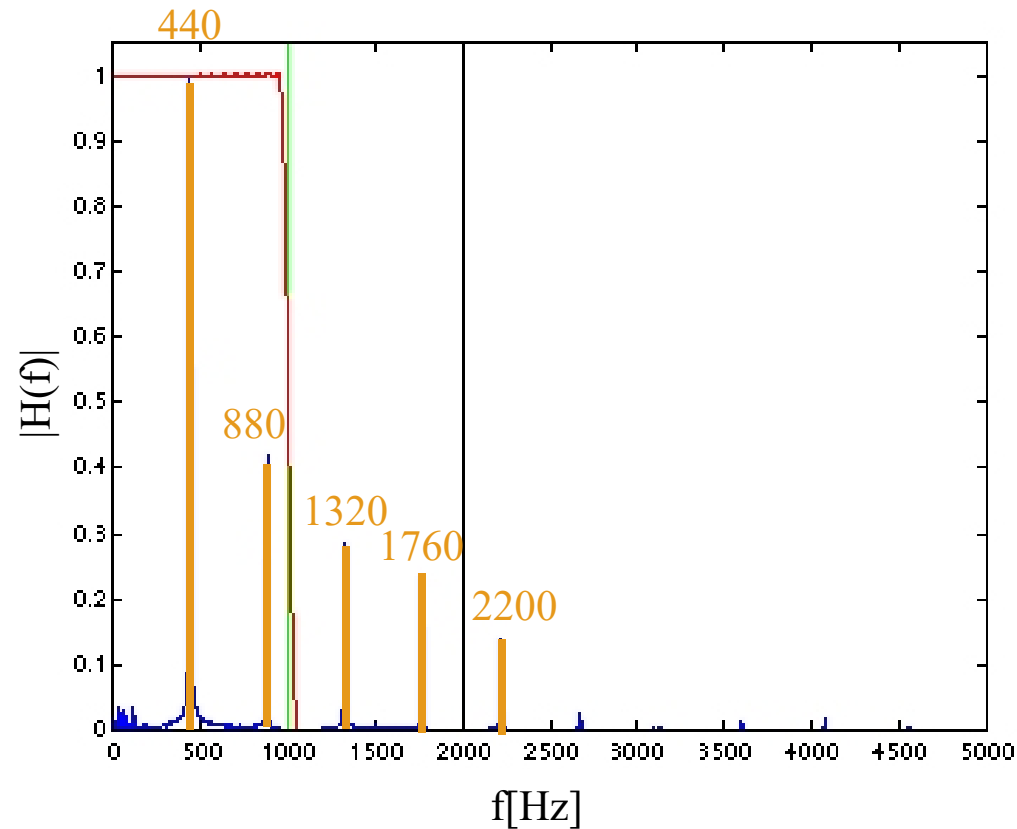
La-Tone (440 Hz) sampled at 44.1 kHz (CD standard)



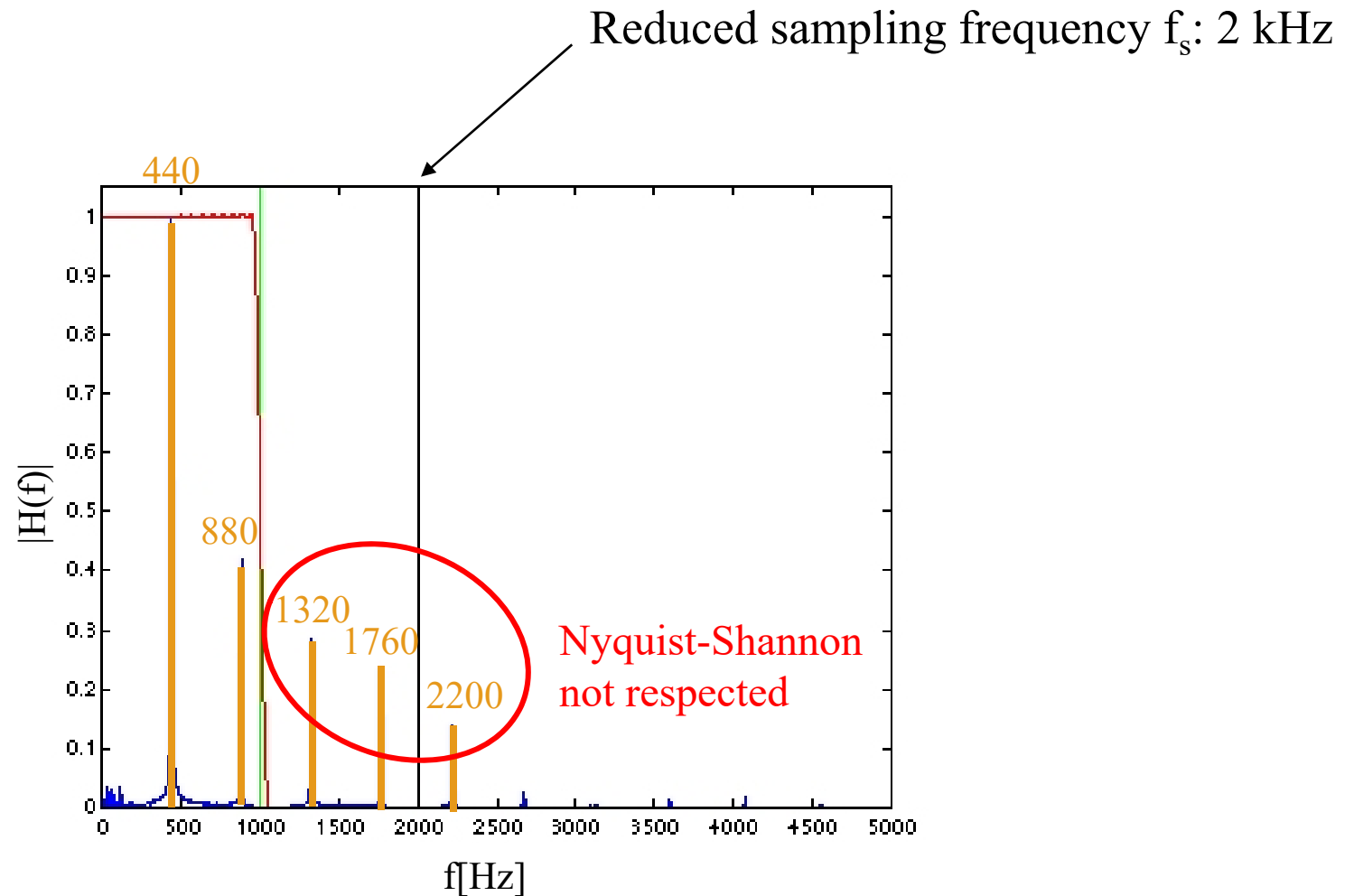
La-Tone (440 Hz) sampled at 2 kHz



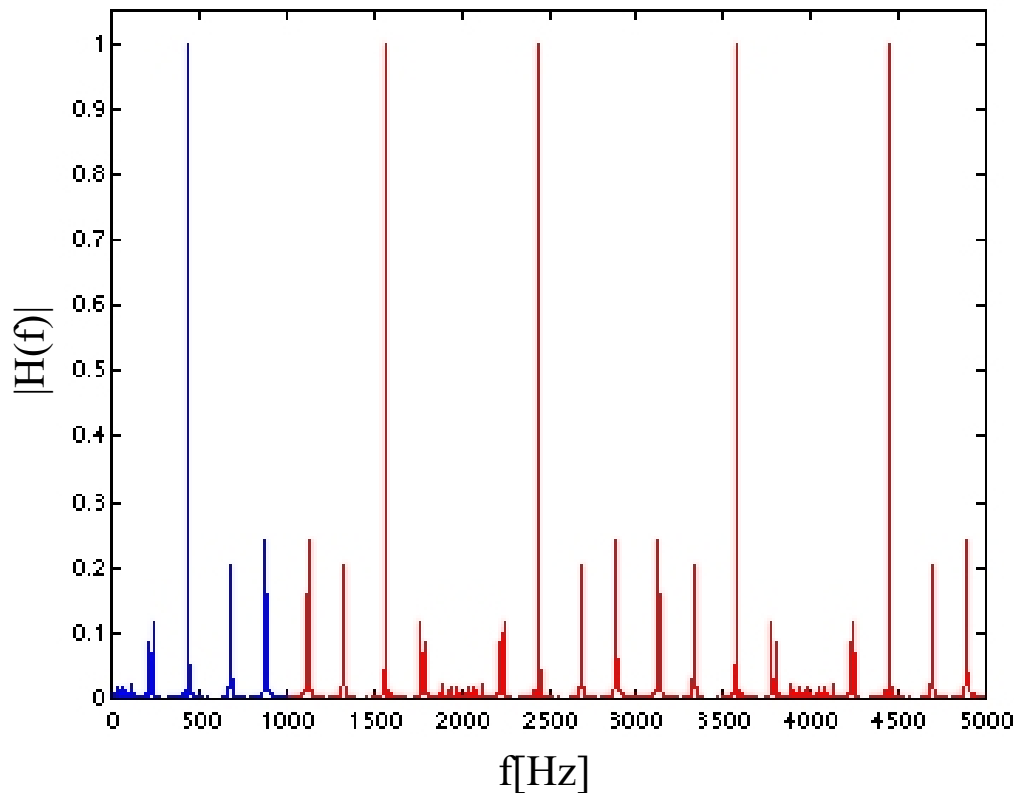
EPFL La-Tone (440 Hz) sampled at 2 kHz



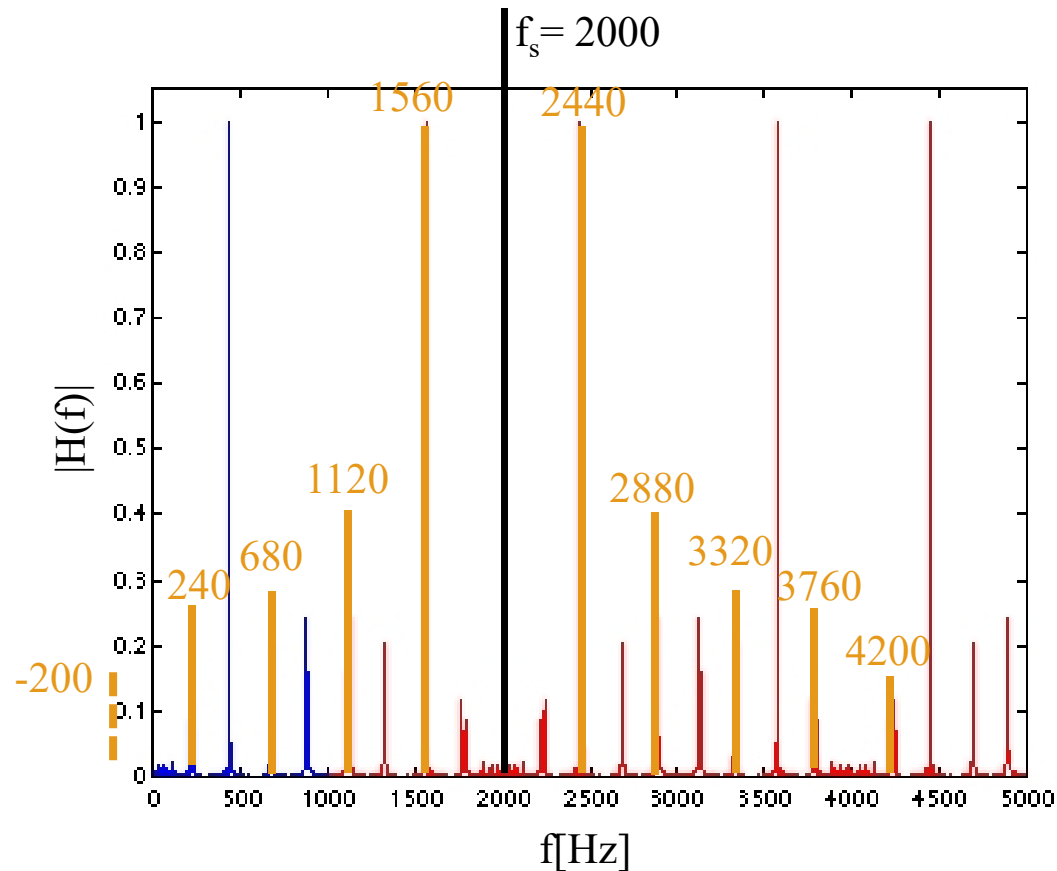
La-Tone (440 Hz) sampled at 2 kHz



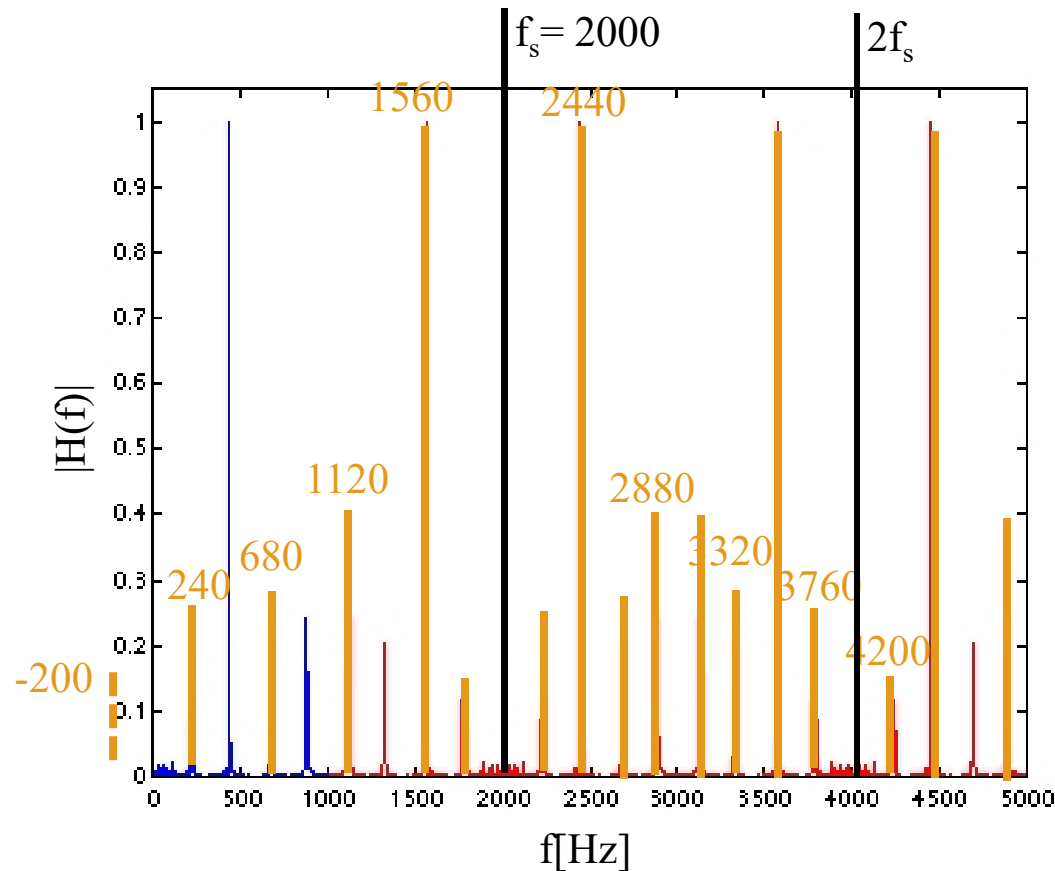
La-Tone (440 Hz) sampled at 2 kHz without filtering



La-Tone (440 Hz) sampled at 2 kHz without filtering



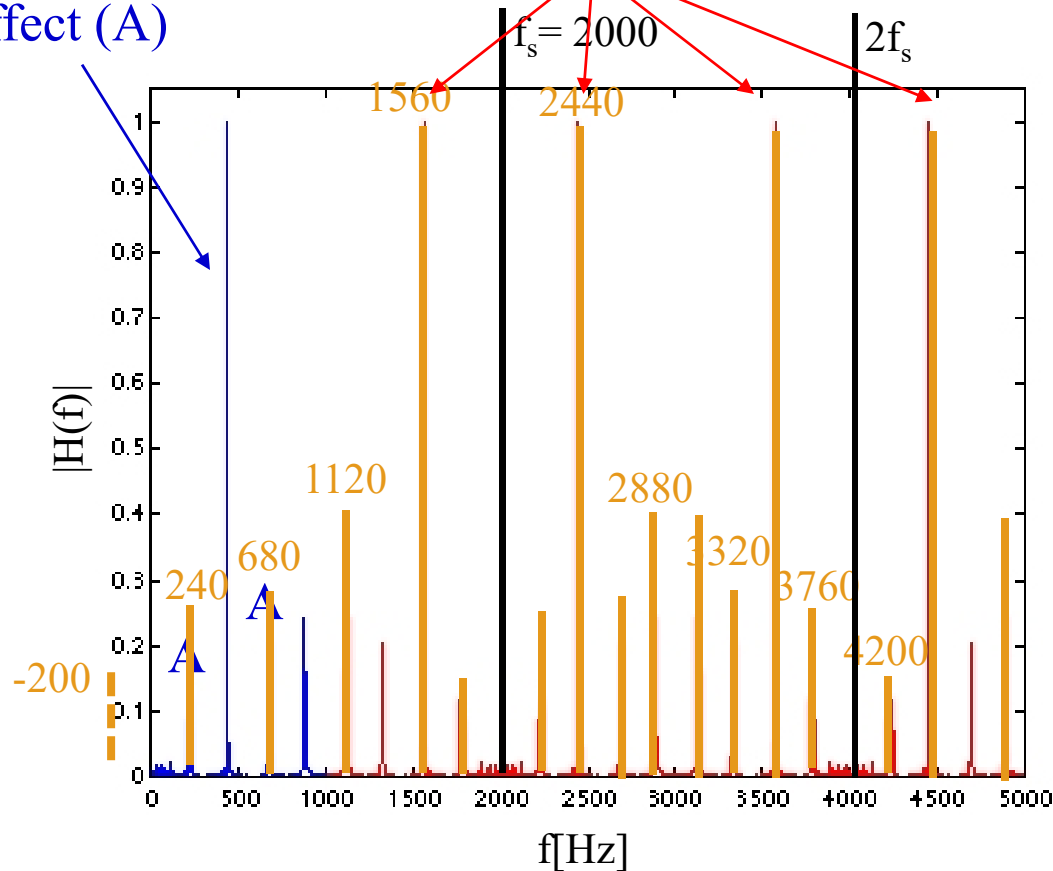
La-Tone (440 Hz) sampled at 2 kHz without filtering



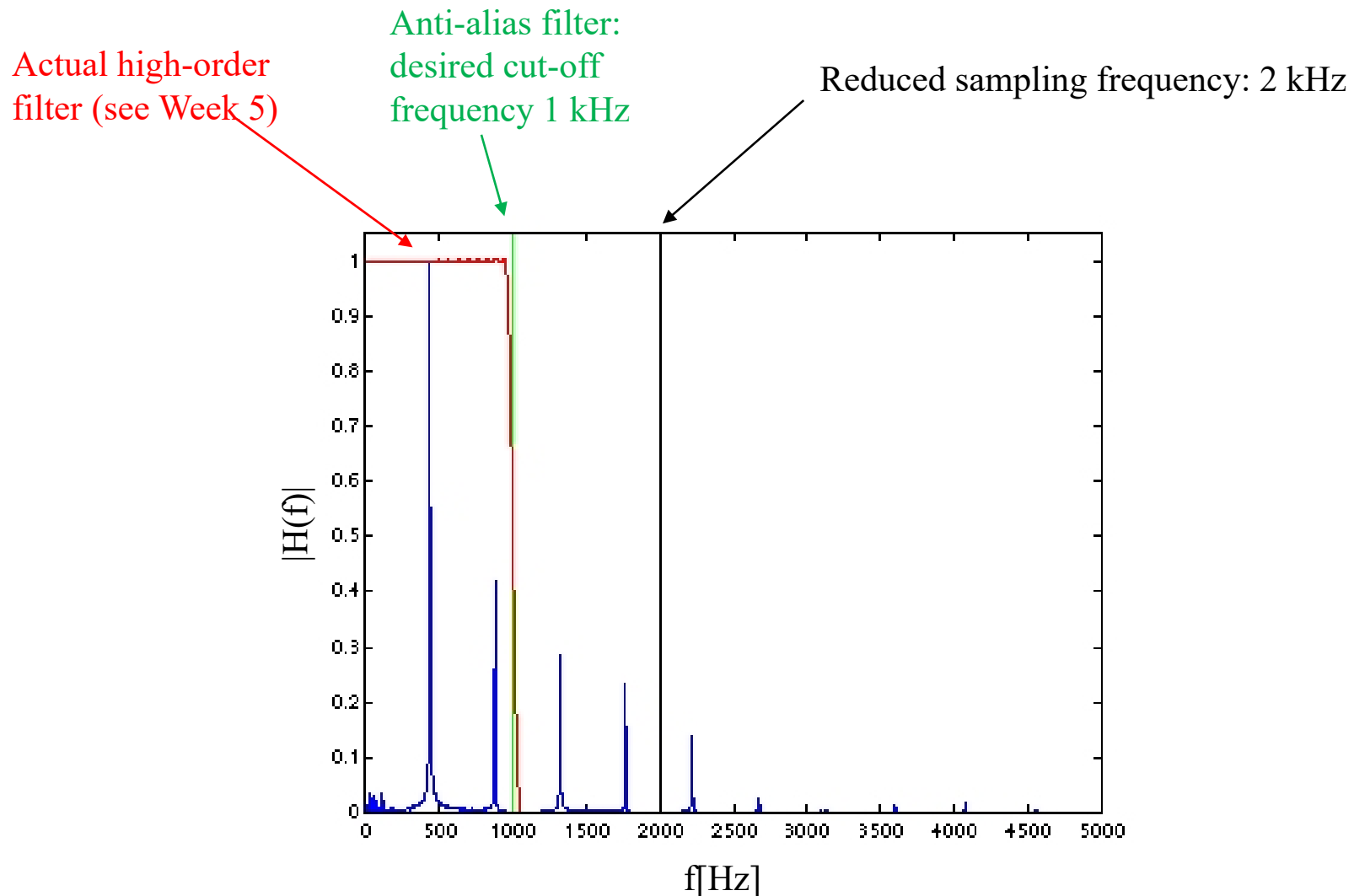
La-Tone (440 Hz) sampled at 2 kHz without filtering

Original signal
with aliasing
effect (A)

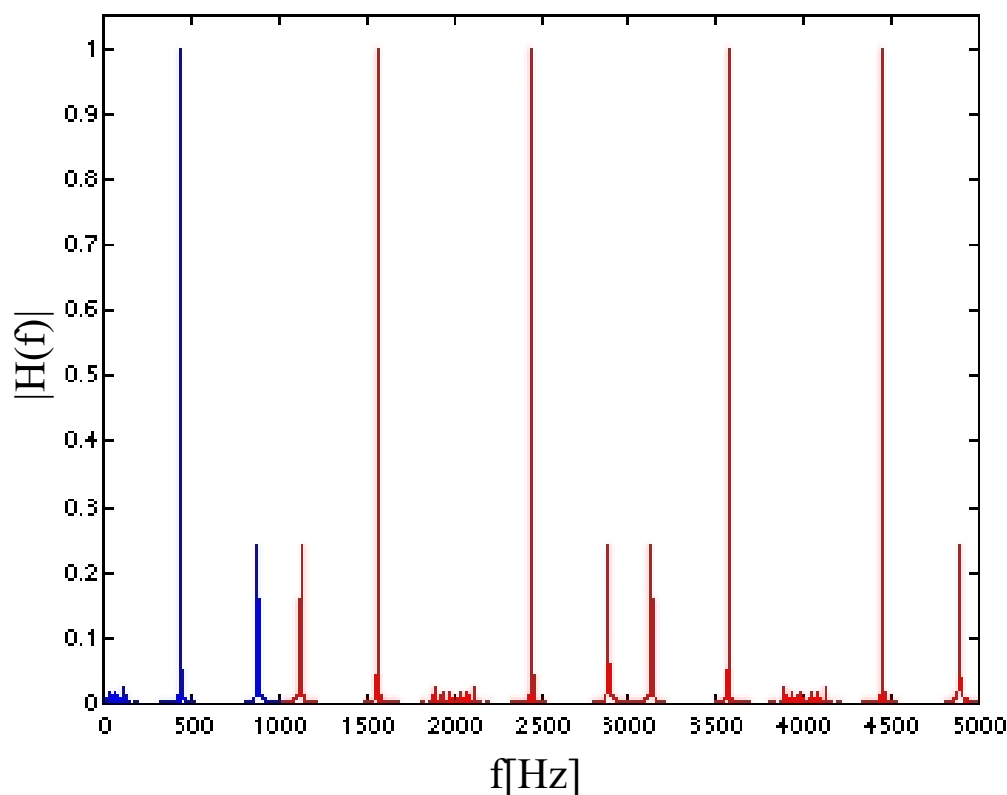
Repeated & shifted spectrum through sampling



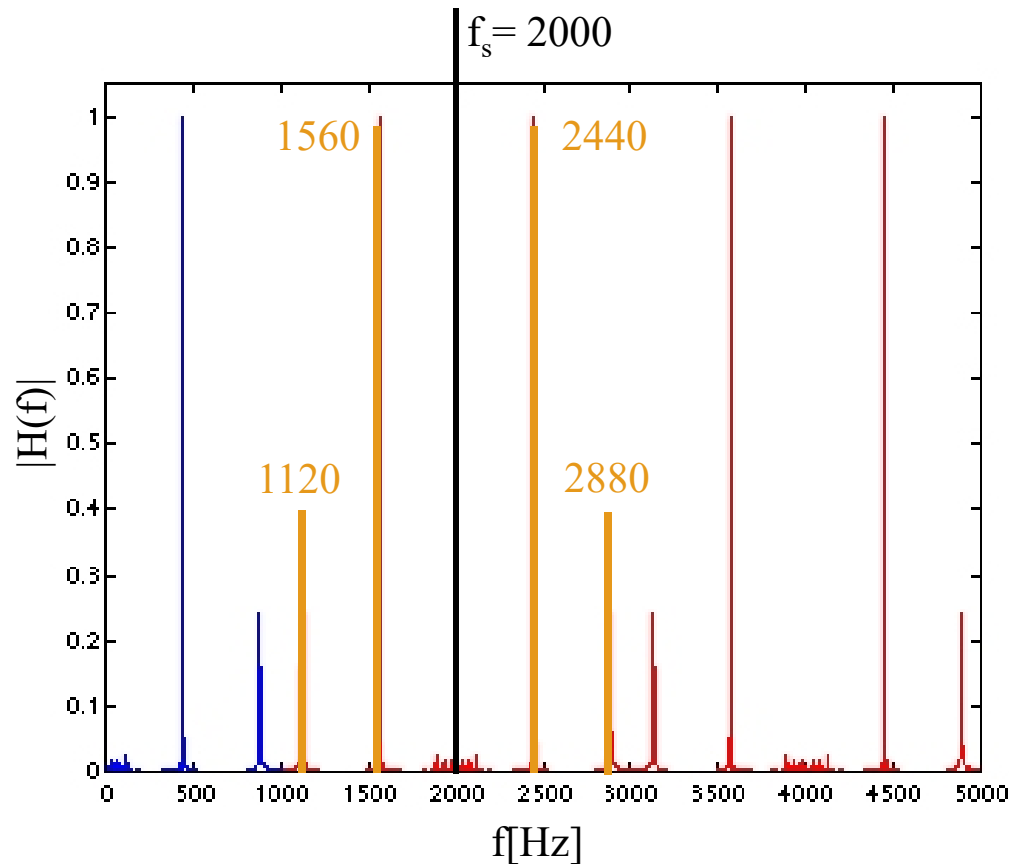
EPFL La-Tone (440 Hz) sampled at 2 kHz



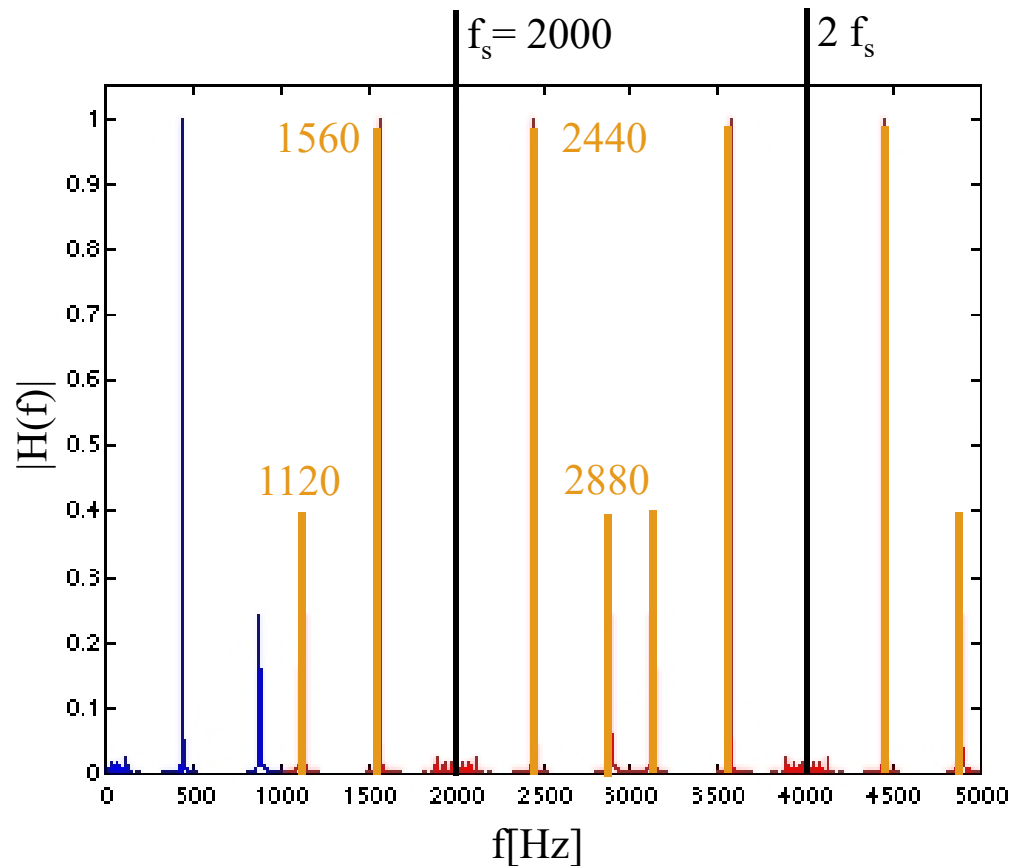
La-Tone (440 Hz) sampled at 2 kHz filtered at 1 kHz



La-Tone (440 Hz) sampled at 2 kHz filtered at 1 kHz



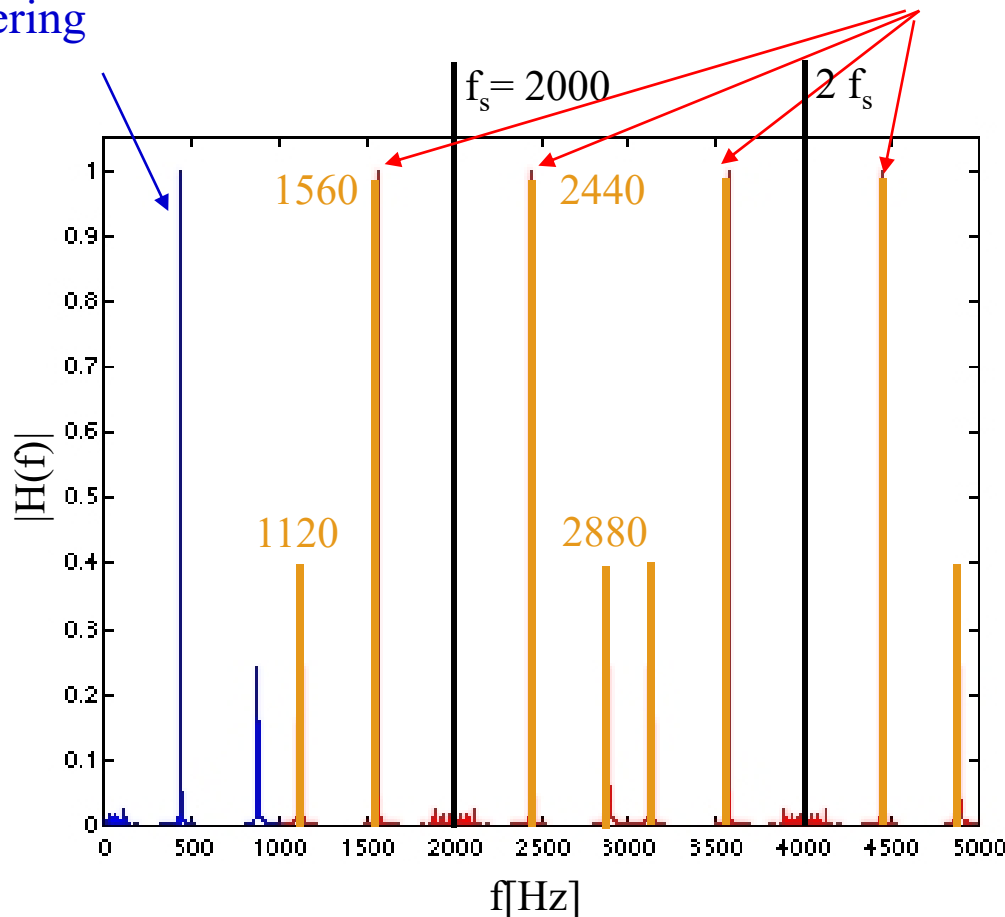
La-Tone (440 Hz) sampled at 2 kHz filtered at 1 kHz



La-Tone (440 Hz) sampled at 2 kHz filtered at 1 kHz

Original signal
without aliases
through anti-alias
filtering

Repeated spectrum, filtered through low-pass anti-alias filter ->
no folding back and therefore no distortion but harmonics above
1 kHz also cut by the anti-alias filter



Aliasing Audio Examples

Original sound



Aliases 4 kHz

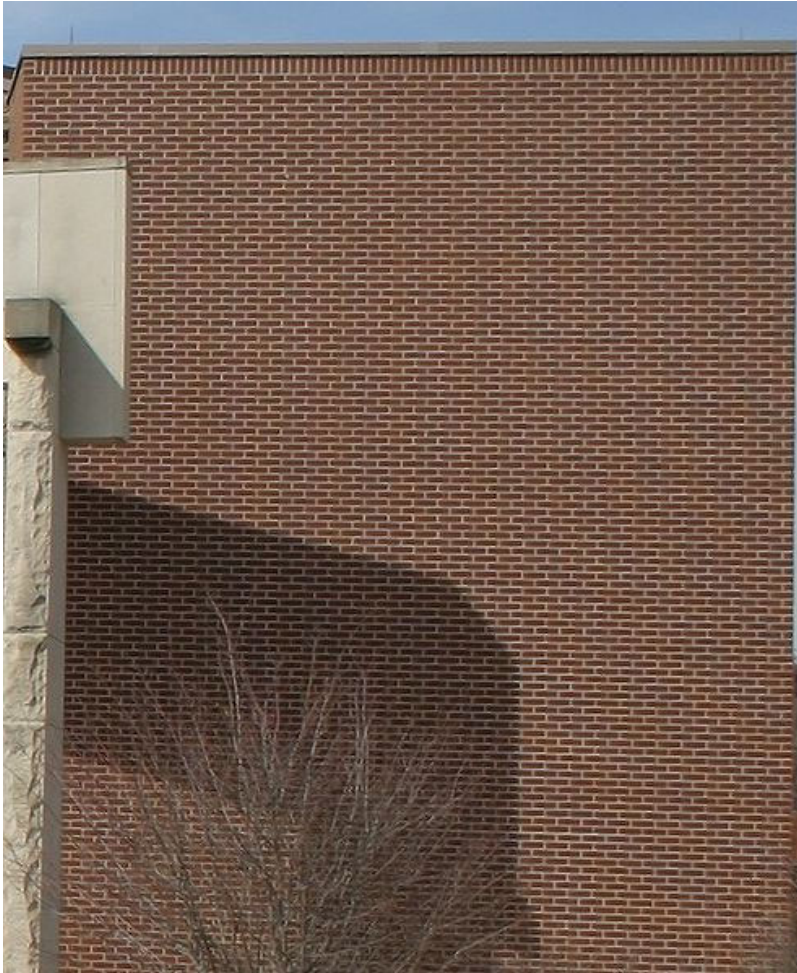


Correct sampling 4 kHz

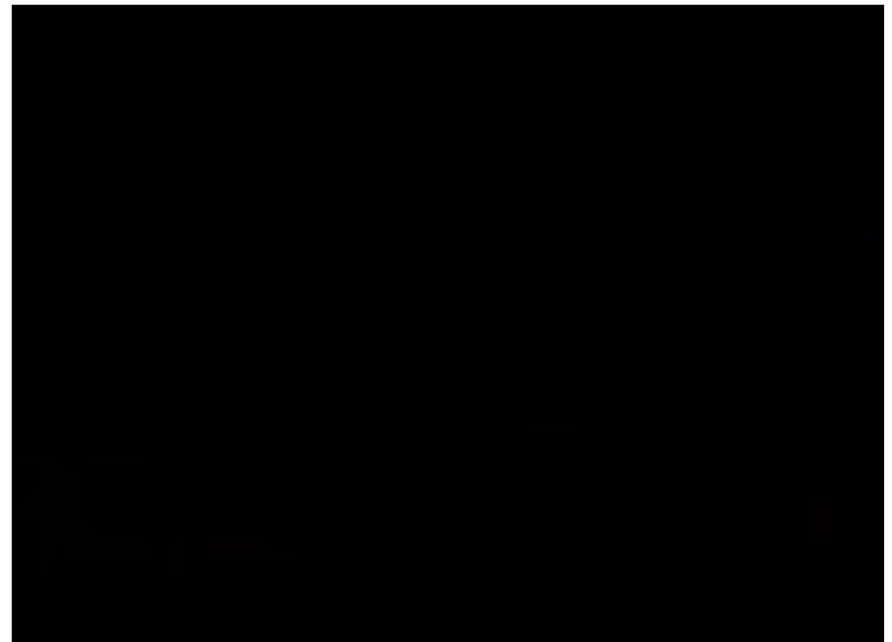


Note: for instance cymbals (8-16 kHz) fully cut off, no matter what

Moiré Pattern



Aliasing Video Examples



<https://www.youtube.com/watch?v=R-IVw8OKjvQ>

<https://www.youtube.com/watch?v=jHS9JGkEOmA>

Conclusion

Take Home Messages

- Sampling: multiplication of the signal with a periodic train of Dirac impulses results in a repeated shifted spectrum in the frequency domain
- Multiple signal reconstructions algorithms are available, more or less computationally expensive; they can all be represented by low-pass filters in the frequency domain
- Aliasing: higher frequency “folded back” on the original spectrum -> prevent proper signal reconstruction
- When you sample a signal ...
 - Make sure you know what the maximum frequency $f_{\max}$ is or enforce it through an anti-alias low-pass filter
 - Make sure you sample at $f_s > 2 f_{\max}$ (Nyquist-Shannon theorem)

Additional Literature – Week 3

- **Books:**
- J. H. McClellan, R. W. Schafer, M. A. Yoder
“DSP First: A Multimedia Approach”, Prentice Hall, 1999.
- A. Oppenheim and A. S. Willsky with S. Nawab, “Signals and Systems”, Prentice Hall, 1997.