Lab 3: Solutions

1. **(Q)** Consider a system where its impulse response (when the impulse is applied at t=0) is given as $h(t) = 3e^{-2t}$ where $t \geq 0$. Calculate the step response of the system by convolving step function $u(t)$ with the impulse response $h(t)$. Remember that the continuous convolution can be calculated by the following equation:

$$(f * g)(t) = \int_{-\infty}^{\infty} f(\tau)g(t-\tau)d\tau$$

Answer: *Let's take the continuous convolution of u(t) with h(t). Pay attention that h(t) = 0 if t<0.*

$$y_u(t) = (h * u)(t) = \int_{-\infty}^{\infty} h(\tau)u(t-\tau)d\tau = \int_{-\infty}^{t} h(\tau)d\tau = \int_{0}^{t} 3e^{-2\tau}d\tau$$ (since h(t) = 0 if t<0)

$$-1.5(e^{-2t}+1) = -1.5e^{-2t} + 1.5$$ where t≥0

2. **(S):** Given that R = 1 Ω and C = 0.1 F, find the impulse response $h(t)$ of the system from the step response $y_u(t)$ given. Plot both the step response and impulse response using Matlab and observe the figure. Can you identify the time-constant from the figure? Why did we use a "Heaviside" function in the code?

Answer: *Due to linearity, the impulse response of the system is the derivative of the step response with respect to time.*

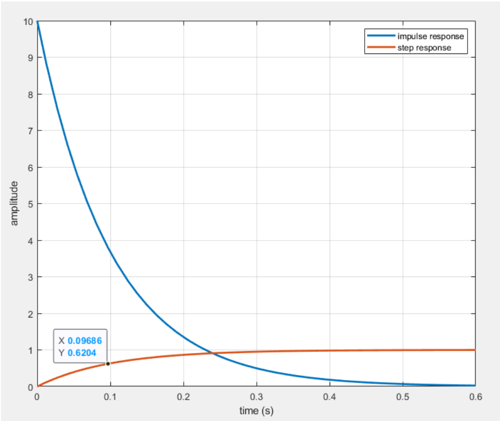
$$y_u(t) = \left(-e^{\left(\frac{-t}{\Gamma}\right)} + 1\right) \cdot u(t) \Rightarrow h(t) = \frac{dy_u(t)}{dt} = \left(\frac{1}{\Gamma} e^{\left(\frac{-t}{\Gamma}\right)}\right) \cdot u(t)$$

The code for MATLAB is given below:

```
% Define constants
R = 1;
C = 0.1;
Gamma = R*C;

 % Define symbolic variables and preferences (time, impulse response,
                                                               output,
% input variables)
syms t h y u
sympref('HeavisideAtOrigin',1);

% Compute step function (input) symbolically (Use heaviside())
u = heaviside(t);
% Compute impulse response symbolically (Use expression for impulse
resp.)
h = 1/Gamma*exp(-1/Gamma*t)*u;
% Compute step response (output) symbolically (Use expression for
step
% resp.)
y = (-exp(-1/Gamma*t)+1)*u;

% Plot
figure
```

```
fplot(h,[0,0.6],'LineWidth',2)
xlabel('time (s)')
ylabel('amplitude')
grid on
hold on
fplot(y,[0,0.6],'LineWidth',2)
legend('impulse response','step response')
xlabel('time (s)')
ylabel('amplitude')
```

The corresponding figure is shown below:



As can be seen from the data point, when the step response is about %63 of the steady-state value, the time gives the time constant which is close to 0.1.

3. (I): Since you know the unit step input and found the impulse response in time domain. You can find the step response by using convolution of both analytically. However, there is no symbolic convolution function in MATLAB. Instead, we can use the definition of continuous convolution to calculate it.

$$y_u(t) = u(t) * h(t) = \int_{-\infty}^{\infty} u(\tau) \cdot h(t-\tau) d\tau$$

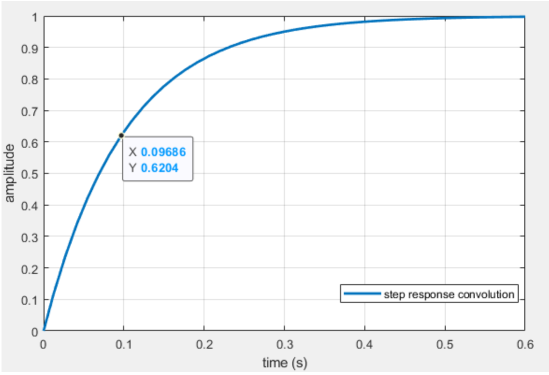
Obtain the step response, plot and compare it with the one that you found in question **1**. Which function did we use to calculate the symbolical integration? Do the overall shape and the time-constant differ?

Answer: *The code for MATLAB is given below: (Note that the code might use previously defined variables)*

```
% Define additional symbolic variables (Convolution variable)
syms tau

% Compute step input for convolution symbolically (Use heaviside)
u_conv = heaviside(tau);
% Compute impulse response for convolution symbolically (Use subs())
h_conv = subs(h,t-tau);
% Compute convolution symbolically (Use int())
conv = int(u_conv*h_conv,tau,-inf,inf);

% Plot
figure
fplot(conv,[0,0.6],'LineWidth',2)
grid on
```

```
legend('step response convolution')
xlabel('time (s)')
ylabel('amplitude')
```

The corresponding figure is shown below:



*As can be seen from the data point, when the step response is about %63 of the steady-state value, the time gives the time constant which is close to 0.1. This is also very similar to question **1**.*

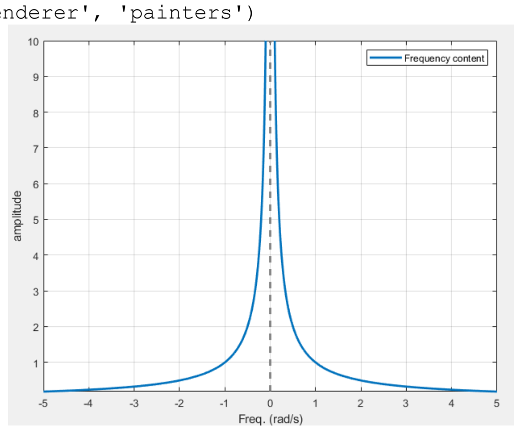
4. **(I):** Analyze the frequency content of the step response y_u (t) by applying the CT Fourier transform in Matlab. Plot the magnitude of the response and observe it. In which frequency region, the frequency content is accumulated? Pay attention to the unit of frequency axis. Is it in Hertz or radians/sec?

Answer: *The code for MATLAB is given below: (Note that the code might use previously defined variables)*

```
%Define additional symbolic variables (Freq variable)
syms w

% Compute Fourier transfom symbolically (Use fourier())
Y_FT = fourier(conv,t,w);

%Plot
figure
fplot(abs(Y_FT));
grid on
legend('Frequency content')
xlabel('Freq. (rad/s)')
ylabel('amplitude')
```

The corresponding figure is illustrated below:

```
set(0, 'DefaultFigureRenderer', 'painters')
```



As can be seen from the figure, the dominant frequency component is at 0 rad/s and decreasing exponentially as frequency increases.

5. **(B):** Given that $h(t) = \left(\frac{1}{\Gamma} e^{\left(\frac{-t}{\Gamma}\right)}\right) \cdot u(t)$ from Question **2**, Compute *U(s)* and *H(s)* by using the definition of Laplace transform or Laplace transform tables in appendix. Find the step response $y_u(t)$ in time domain by using the definition of inverse Laplace transform or inverse Laplace transform tables.

Answer: *For this question we will use Laplace transform tables. From Table 9.2, 2nd entry,*

$$u(t) = unitstep \Rightarrow U(s) = \frac{1}{s} \qquad \text{where } \Re\{s\} > 0$$

From Table 9.2, 6th entry,

$$h(t) = \left(\frac{1}{\Gamma} e^{\left(\frac{-t}{\Gamma}\right)}\right) \cdot u(t) \Rightarrow H(s) = \frac{1}{\Gamma} \cdot \frac{1}{s+\frac{1}{\Gamma}} \qquad \text{where } \Re\{s\} > \frac{-1}{\Gamma}$$

Therefore, Laplace transform of x(t) can be written as:

$$Y(s) = H(s)U(s) = \frac{1}{\Gamma} \cdot \frac{1}{s^2 + \frac{s}{\Gamma}} = \frac{A}{s + \frac{1}{\Gamma}} + \frac{B}{s}$$

where we need to find A and B in order to utilize Laplace tables. When we equate LHS to RHS, Y(s) can be found as:

$$A = -1, B = 1 \Rightarrow Y(s) = \frac{-1}{s+\frac{1}{\Gamma}} + \frac{1}{s} \qquad \text{where } \Re\{s\} > 0$$

By using Table 9.2, 2nd and 6th entries, x(t) can be written as:

$$y_u(t) = \left(-e^{\left(\frac{-t}{\Gamma}\right)} + 1\right) \cdot u(t)$$

which is consistent with question ***2.***

6. **(I):** Given that impulse response $h(t) = \left(\frac{1}{\Gamma} e^{\left(\frac{-t}{\Gamma}\right)}\right) \cdot u(t)$ from Question **2,** apply the same procedure explained in question **5** in Matlab symbolically, instead of using tables, and find the step response $y_u(t)$. Which functions did you use to take the Laplace and inverse Laplace transform symbolically? Plot and compare the result with the one you found in Questions **2** and **3**.

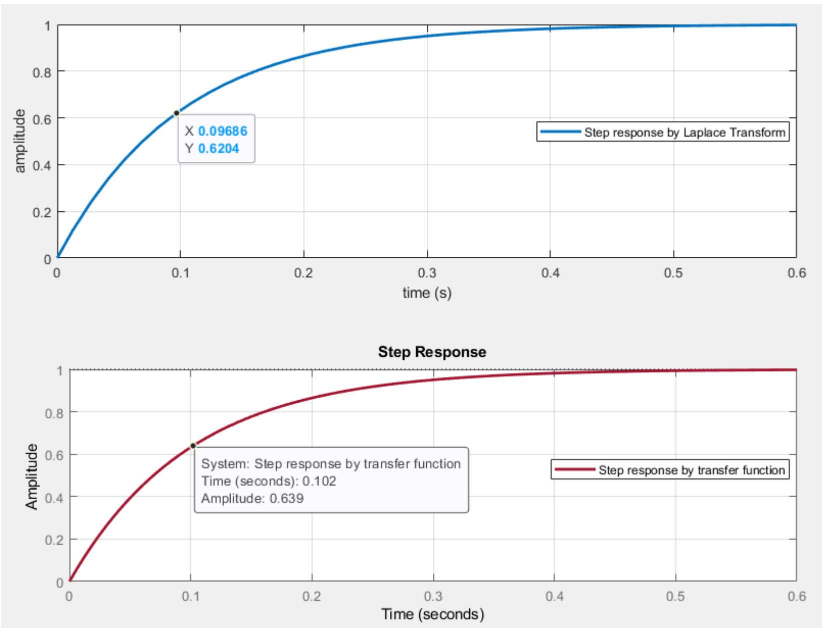
Answer: *The code for this question will be given together with the one for question 7 in order to better compare the results.*

7. **(I):** Another way to obtain step response is to use `tf()` function together with the transfer function representation you have found. Observe $H(s)$ and represent it in Matlab using `tf()` function. Use the `step()` function together with the resulting transfer function to obtain the plot of step response. Plot and compare the result with the ones you found in Questions **2, 3,** and **6**. Do the overall shape and time-constant differ?

Answer: *The MATLAB code for questions 6 and 7 are given together as follows: (Note that the code might use previously defined variables)*

```
% Define additional symbolic variables (Laplace variable)
syms s
% (Q6)
% Compute Laplace transform of input symbolically (Use laplace())
U_LT = laplace(u,t,s);
% Compute Laplace Transform of impulse response symbolically (Use
laplace())
H_LT = laplace(h,t,s);
% Compute step response in time domain symbolically (Use ilaplace())
y_2 = simplify(ilaplace(U_LT*H_LT,t));

% (Q7)
% Compute transfer function (Use tf())
H_LT_tf = tf(10,[1,10]);

figure
subplot(2,1,1)
fplot(y_2,[0,0.6],'LineWidth',2);
grid on
xlabel('time (s)')
ylabel('amplitude')
legend('Step response by Laplace Transform');
subplot(2,1,2)
% Plot step response
step(H_LT_tf,0.6);
grid on
legend('Step response by transfer function');
```

The corresponding figures are given below:



As can be observed from the figures, both time constants are around 0.1. Hence, they are very similar to previously drawn figures.